Vol. 29, No. 2, Nov.-Dec., 1955

MATHEMATICS magazine

MATHEMATICS MAGAZINE

Formerly National Mathematics Magazine, founded by S. T. Sanders.

Address editorial correspondence to Glenn James, special papers to the editors of the departments for which they are intended, and general papers to a member of the executive committee.

Manuscripts should be typed on 8½" x 11" paper, double-spaced with 1" margins. We prefer that, in technical papers, the usual introduction be preceded by a *Foreword* which states in simple terms what the paper is about.

The Mathematics Magazine is published at Pacoima, California by the managing editor, bi-monthly except July-August. Ordinary subscriptions are 1 yr. $3.00; 2 yrs. $5.75; 3 yrs. $8.50; 4 yrs. $11.00; 5 yrs. $13.00. Sponsoring subscriptions are $10.00; single copies 65¢. Reprints, bound, ½¢ per page plus 10¢ each, (thus 25 ten page reprints would cost $1.25 plus $2.50 or $3.75) provided your order is placed before your article goes to press.

Subscriptions are related correspondence should be sent to Inez James, 14068 Van Nuys Blvd., Pacoima, California.

Advertisers should contact The Lees Agencies, P.O. Box 906, Hollywood, 28, California.

Entered as second-class matter March 23, 1948 at the Post Office, Pacoima, California under act of congress of March 8, 1876.

MATHEMATICS MAGAZINE

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OUR CONTRIBUTORS

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Jerome Hines is a distinguished singer at the Metropolitan Opera and is often heard on radio and television. A graduate of the University of California, Los Angeles, with a major in mathematics, he is still keenly interested in the subject and has contributed several articles to this Magazine.

Robert E. Horton, Instructor at Los Angeles City College, attended both the University of California, Los Angeles (A.B.'36, M.A.'47) and the University of Southern California. During World War II he served as Director of Ground Training at an Air Force School and later with the 15th Air Force in Italy. Since 1953 he has been Problem Editor of this Magazine. His article, "Geometry in General Education," appeared in the September-October issue.

Robert C. James, Associate Professor of Mathematics, Haverford College, is the son of our Managing Editor, Glenn James. A graduate of the University of California, Los Angeles (B.A.'40) and of the California Institute of Technology (Ph.D.'46) Dr. James was a Benjamin Pierce Instructor at Harvard in 1946-47 and taught at the University of California from 1947-51. His fields of research include linear spaces and topological groups. His expository article on combinatorial topology was published in the September-October issue.

SOME ASSOCIATIVE OPERATIONS ON INTEGERS

J. Lambek and L. Moser

In this note we exhibit some simple (1-1) correspondences between the non-negative integers and certain sets of pairs of integers. These are used to establish the associativity of certain binary operations on the integers.

Let $[x]$ be the largest integer not exceeding x and $\{x\} = [x + 1/2]$ the integer nearest to x. Consider the mapping:

(1) $$n \to (\{\sqrt{n}\},\ \ n - \{\sqrt{n}\}^2).$$

For $n = a^2 + b$, with $-a < b \leq a$, we have $\{\sqrt{n}\} = a$ and $n - \{\sqrt{n}\}^2 = b$. Conversely, if a and b are thus given in terms of n, then $a^2 + b = n$. Hence (1) has the inverse mapping

(2) $$(a, b) \to a^2 + b.$$

Thus (1) and (2) provide a (1-1) correspondence between the non-negative integers and the pairs of integers (a, b) with $-a < b \leq b$. This correspondence is illustrated in figure 1.

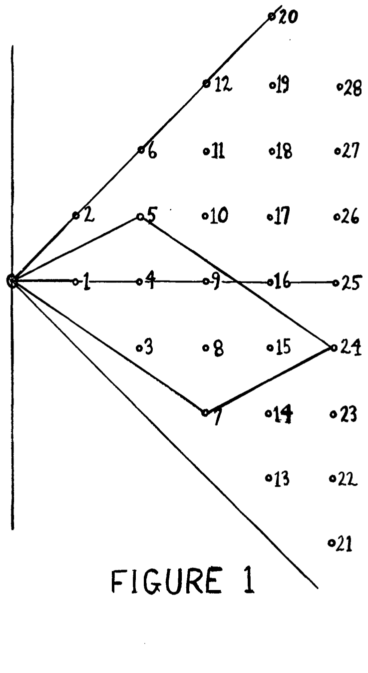

FIGURE 1

Theorem 1: *The binary operation* o *defined on the non-negative integers by*

(3) $$m \circ n = m + n + 2\{\sqrt{m}\}\{\sqrt{n}\}$$

is associative, i.e. $(1 \circ m) \circ n = 1 \circ (m \circ n)$.

Proof. Given m and n, combine them as follows: First, find the corresponding pairs of integers by (1). Next, add these pairs vectorially. Finally, use (2) to find the integer corresponding to this vector sum. The combination of 5 and 7 to yield 24 is illustrated in figure 1. This mode of combination is defined and associative, because vector addition of pairs (a,b) in the region $-a < b \leq a$ is defined and associative. Explicitly, we have by (1)

$$m \to (\{\sqrt{m}\},\ m - \{\sqrt{m}\}^2),\quad n \to (\{\sqrt{n}\},\ n - \{\sqrt{n}\}^2).$$

The sum of these two vectors is

$$(\{\sqrt{m}\} + \{\sqrt{n}\},\ m + n - \{\sqrt{m}\}^2 - \{\sqrt{n}\}^2),$$

which is mapped by (2) onto

$$(\{\sqrt{m}\} + \{\sqrt{n}\})^2 + m + n - \{\sqrt{m}\}^2 - \{\sqrt{n}\}^2 = m \circ n.$$

Thus the operation described is the one given by (3), and the theorem is proved.

We may define "primes" under the operation (3) as positive integers p for which $p = m \circ n$ implies $m = p$ or $n = p$. It should be clear from figure 1 that the "primes" are 0,2, and the interers of the form $n^2 + n + 1$ for $n \geq 0$. However, there is no unique factorization into primes, as for example $9 = 1 \circ 1 \circ 1 = 2 \circ 3$. More generally, $(2n + 1)^2$ can be factored into $2n + 1$ factors 1 or into n factors 2 and one factor $n^2 + n + 1$.

We proceed to generalize theorem 1. Let $f(0) = 0$, $f(1)$, $f(2), \cdots$ be an increasing sequence of non-negative integers. Write $\Delta f(m) = f(m + 1) - f(m)$. Let $g(n)$ be the largest m for which $f(m) \leq n$, then $g(n)$ is defined for all $n \geq 0$, and $f(g(n))$ is the largest element of the sequence $f(m)$ not exceeding n. For example, if $f(m) = m^2$, then $g(n) = [\sqrt{n}]$, and $f(g(n))$ is the largest square not exceeding n. Given any non-negative integer n, we can express it uniquely in the form

(4) $$n = f(g(n)) + r(n),$$

where

(5) $$0 \leq r(n) < \Delta f(g(n)).$$

Now consider the mapping

(6) $$n \to (g(n),\ r(n)).$$

This has as its inverse

(7) $$(a,b) \to f(a) + b$$

where

(8) $$0 \le a, \quad 0 \le b < \Delta f(a).$$

Theorem 2: *If $f(n)$ satisfies the inequality*

(9) $$\Delta f(a) + \Delta f(b) \le \Delta f(a+b) + 1$$

then the operation

(10) $$m \circ n = m + n + f(g(m) + g(n)) - f(g(m)) - f(g(n))$$

is associative.

Proof. Given m and n, combine them as follows: First, find the corresponding pairs of integers, using (6). Next, add these vectorially to obtain

(11) $$(g(n) + g(n), \quad r(m) + r(n))$$

We wish to show that this is a point in the region defined by (8), i.e.

$$r(m) + r(n) < \Delta f(g(m) + g(n)).$$

First recall that $g(m)$ is the largest k for which $f(k) \le m$, so that

$$f(g(m) + 1) = f(k+1) \ge m + 1.$$

Hence by (4).

$$r(m) = m - f(g(m)) \le f(g(m) + 1) - 1 - f(g(m)) = \Delta f(g(m)) - 1$$

so that

$$r(m) + r(n) \le \Delta f(g(m)) + \Delta f(g(n)) - 2 \le \Delta f(g(m) + g(n)) - 1$$

by (9). Thus (11) yields a point in the region (8), and we may find its image under (7), namely

$$f(g(m) + g(n)) + m - f(g(m)) + n - f(g(n)) = m \circ n,$$

according to (10). This operation is therefore isomorphic to vector-addition, hence associative, as was to be proved.

Theorem 1 may be obtained as a special case from theorem 2 by considering $f(n) = n^2 - n + 1$ for $n > 0$, $f(0) = 0$, $g(n) = \{\sqrt{n}\}$. However, the proof of theorem 2 does not reduce to the above proof of theorem 1. Another special case is

$$f(n) = n^2, \quad g(n) = [\sqrt{n}], \quad m \circ n = m + n + 2[\sqrt{m}][\sqrt{n}].$$

The associative operation $m \circ n = m + n + mn$, though similar in appearance to the above and to (3), arises in a different way. All the operations considered so far are commutative. The operation suggested by $27 \circ 356 = 27356$, defined on the positive integers, is associative but not commutative. An even simpler operation satisfying these conditions is $m \circ n = n$. It turns out that all the operations mentioned in this paragraph are covered by the formula

$$m \circ n = mf(n) + n,$$

where $f(n)$ satisfies the functional equation

$$f(m\, f(n) + n) = f(m)\, f(n).$$

Thus to obtain the three cases mentioned, put

$$f(n) = n + 1, \qquad f(n) = 10^{[\log_{10} n]+1} \qquad f(n) = 0$$

respectively.

Research Institute Canadian Mathematical Congress.

Polish Academy of Sciences in Tribute to Einstein

The memory of Prof. Albert Einstein has been honored in a special session devoted to his scientific achievements by the executive body of the Polish Academy of Sciences. Research workers and university teachers of the physical sciences throughout the country joined in the tribute.

Several of Einstein's Polish collaborators spoke of his enormous contributions to the progress of science. Prof. Leopold Infeld, outstanding theoretical physicist who worked with Einstein in Europe and the United States, read a paper on "Fifty Years of the Theory of Relativity." Professor Einstein, he said, "fought passionately for many years against the use of atomic energy for war purposes. Today he is no longer among us but I express the hope that his science as well as the ideals of peace for which he labored may inspire the minds and hearts of mankind forever."

Prof. Michal Smialowski, secretary of the Academy's section on the physical sciences, described Einstein "as the greatest physicist of our time and a noble example of the scientist devoted to the cause of peace." A similar theme was stressed by Prof. Stanislaw Loria in his paper on Einstein's contribution to the development of quantum physics.

The commemorative meeting was held May 18.

ON THE CONVERGENCE RATE OF AN ITERATIVE PROCESS

Robert T. Gregory

There are many problems in applied mathematics for which approximate solutions are easily obtained by iterative methods. Consider the equation

(1) $$\phi_{n+1} = L\phi_n$$

where L is the ''improvement operator'' of an iteration process. Given an initial estimate ϕ_0 of the solution ϕ, a sequence $\{\phi_n\}$ of improved estimates can be formed by repeated application of the operator L. The limit of the sequence will be the solution ϕ.

To determine the usefulness of an iteration scheme we need to know something about its rate of convergence. In order to study the convergence of the sequence $\{\phi_n\}$ we go instead to the null sequence $\{e_n\}$ where e_n is the error in the estimate ϕ_n of ϕ at the nth stage, i.e.

(2) $$e_n = \phi_n - \phi.$$

If we restrict ourselves to linear operators L then it can be shown that e_n satisfies (1). From equation (2)

$$Le_n = L(\phi_n - \phi)$$

$$= \phi_{n+1} - \phi.$$

Thus

(3) $$e_{n+1} = Le_n$$

and (3) becomes the error recurrence relation.

Frankel[1] describes a well-known method for estimating the size of the error at any stage, i.e., a method for deciding when a process has " converged" in the sense that the error is as small as we desire. This method depends on the fact that for a suitably restricted class of operators we can obtain an expansion of the initial error e_0 in terms of the eigenfunctions of the operator L, i.e.,

(4) $$e_0 = \sum_i a_i y_i$$

where the eigenfunctions y_i satisfy the equation

(5) $$Ly_i = \lambda_i y_i$$

and y_i is the eigenfunction corresponding to the eigenvalue λ_i.

Using (3), (4) and (5) we find

$$\begin{aligned} e_1 &= Le_0 \\ &= L\sum_i a_i y_i \\ &= \sum_i a_i y_i \lambda_i . \end{aligned}$$

Thus the error after n iterations becomes

$$\begin{aligned} e_n &= L^n e_0 \\ &= L^n \sum_i a_i y_i \\ &= \sum_i a_i y_i \lambda_i^n \end{aligned}$$

and in general the process will converge only if $|\lambda_i| < 1$ for all i. In fact the ultimate convergence rate is determined by the magnitude of the largest eigenvalues.

Now consider the case where the operator L is an mth order matrix and the solution ϕ consists of m discrete functional values. In this case both ϕ_n and e_n become m-tuples and can be considered as vectors in an m-dimensional vector space or as points in an $(m-1)$-dimensional projective space. If we choose the latter viewpoint equations (1) and (3) can be interpreted as collineations in a projective space and the discussion of the eigenfunctions of the operator L becomes a discussion of the fixed points of the collineation. The condition that e_0 shall have an expansion in terms of the eigenfunctions of L becomes the condition that there shall be m linearly independent fixed points of collineation, i.e., that there are enough to span the space.

As an example, Shortly, Weller, and Fried[2] have written a paper describing a numerical solution of Laplace's equation in two independent variables,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

The solution $V(x,y)$ could be obtained by solving the differential equation for a particular set of boundary values. However, in many cases an analytic solution is not practical and a numerical solution is sought. The method described in the above paper is to replace the differential equation by a finite difference equation applied at the points of a rectangular lattice imposed on the region under consideration. The solution is then a set of m functional values, i.e., values of $V(x_i, y_i)$ where x_i, y_i are the coordinates of the lattice points. These lattice points can be ordered and so the m functional values form an m-tuple and can be considered as a point in $(m-1)$-dimensional projective space. Several iteration schemes are described whereby an initial set of functional values $V_0^{(i)}$, $i = 1,2,\cdots,m$, is assigned and these values are improved by the matrix operator M

$$V_{n+1}^{(i)} = MV_n^{(i)} \tag{6}$$

Equation (6) then is a collineation in the projective space.

As was stated earlier it is of utmost importance in the study of the convergence rate of an iterative process to be able to express the initial error e_0 as a linear combination of the eigenfunctions (i.e., in this case, fixed points) of the operator M. *Thus we need to know the conditions under which this is possible.*

Hodge and Pedoe[3] discuss the canonical forms of collineations, their latent roots (eigenvalues), and the "fundamental spaces" of fixed points associated with each latent root. They point out that for collineations over an algebraically closed field matrices of non-singular collineations can be reduced to the form

$$C = \begin{pmatrix} C_{e_1}(\alpha_1) & 0 & \cdot & 0 \\ 0 & C_{e_2}(\alpha_2) & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & C_{e_k}(\alpha_k) \end{pmatrix} \tag{7}$$

where the λ-matrix $C-\lambda I$ has elementary divisors

$$(\lambda - \alpha_1)^{e_1}, \cdots, (\lambda - \alpha_k)^{e_k}$$

and $C_e(\alpha)$ is the $e \times e$ matrix

$$C_e(\alpha) = \begin{pmatrix} \alpha & 1 & 0 & \cdot & 0 \\ 0 & \alpha & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \alpha & 1 \\ 0 & \cdot & \cdot & 0 & \alpha \end{pmatrix} \tag{8}$$

We can divide the matrices C into two classes: (*a*) those matrices for which each submatrix $C_e(\alpha)$ consists of the single element α, and (*b*) those matrices for which at least one submatrix $C_e(\alpha)$ does not consist of the single element α. Matrices in class (*a*) are called diagonalizable by B. E. Mitchell[4] and diagonable by M. P. Drazin.[5]

They have Segre symbols consisting only of ones (e.g. [1,1,1,1] and [(1,1),1,1]) and Mitchell concludes his paper by stating a theorem of Weierstrass (1861) that "a matrix is diagonalizable if and only if it has only simple elementary divisors".

Now the condition that permits the desired expansion of e_0 in terms of the fixed points of a collineation is that the matrix of the collineation shall belong to class (*a*). Since an *m* th order diagonalizable matrix has *m* latent roots there will be *m* linearly independent fixed points. These furnish the basis for the expansion. If two or more latent roots are equal there will be a "fundamental space" of fixed points corresponding to the multiple latent root.

The next question that arises is whether we can apply this method to matrix operators in class (*b*). A matrix of this type will have at least one of its submatrices $C_e(\alpha)$ of the form (8). Corresponding to the latent root α there is only one fixed point and so there will not be enough linearly independent fixed points to span the space. However, Wedderburn[6] points out that for an elementary divisor

$$(\lambda - \alpha)^e$$

there is associated a set of linearly independent points $y_1, y_2, \cdots, y_e$ such that

$$Cy_1 = \alpha y_1$$

$$Cy_2 = \alpha y_2 + y_1$$

$$\cdots$$

$$Cy_e = \alpha y_e + y_{e-1}$$

Notice that y_1 is the only point of the set which is actually a fixed point of

$$x = Cy,$$

but there are enough linearly independent points associated with the latent root α to enable one to form a basis for the space.

In the appendix to their paper[2] Shortly, Weller, and Fried mention this set of linearly independent points discussed above. They point out that in many instances in practice one can obtain desirable results, even if the "improvement operator" is not a diagonalizable matrix, by expanding e_0 in terms of a basis which includes a set of linearly independent points of the type mentioned. Best results are obtained if the latent root α, with which the system of points $y_1, \cdots, y_e$ is associated, is one of the smallest latent roots. In this connection they state that "it seems likely from geometrical and asymptotic considerations that the largest roots··· belong to step-matrices of order one." They also inply that, in practice, only the very small roots belong to larger order stepmatrices and "they are quickly wiped out by the iteration and do not cause trouble."

To illustrate, suppose the matrix operator has the canonical form

$$C = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 1 \\ 0 & 0 & 0 & \gamma \end{bmatrix}$$

Then there are three latent roots α, β, and γ. Corresponding to the latent roots α and β we have the fixed points y_1 and y_2 respectively. Likewise, corresponding to the latent root γ we have the fixed point y_3 and the linearly independent point y_4 which satisfy the equations

$$Cy_3 = \gamma\, y_3$$
$$Cy_4 = \gamma\, y_4 + y_3\ .$$

If we let

$$e_0 = a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4$$

then

$$\begin{aligned} e_1 &= Ce_0 \\ &= a_1\alpha y_1 + \alpha_2\beta y_2 + a_3\gamma y_3 + a_4(\gamma y_4 + y_3) \\ &= a_1\alpha y_1 + a_2\beta y_2 + (a_3\gamma + a_4)y_3 + a_4\gamma y_4 \end{aligned}$$

Likewise

$$\begin{aligned} e_2 &= C^2e_0 \\ &= a_1\alpha^2 y_1 + a_2\beta^2 y_2 + (a_3\gamma^2 + a_4\gamma)y_3 + a_4\gamma(\gamma y_4 + y_3) \\ &= a_1\alpha^2 y_1 + a_2\beta^2 y_2 + (a_3\gamma^2 + 2a_4\gamma)y_3 + a_4\gamma^2 y_4 \end{aligned}$$

and finally

$$\begin{aligned} e_n &= C^ne_0 \\ &= a_1\alpha^n y_1 + a_2\beta^n y_2 + (a_3\gamma^n + na_4\gamma^{n-1})y_3 + a_4\gamma^n y_4. \end{aligned}$$

It is clear that the presence of the term $n a_4 \gamma^{n-1}$ will slow up the convergence somewhat. However, if γ is the smallest latent root the first two terms of the sum eventually will make the greatest contribution and the size of the error after n iterations will depend only on α^n and β^n.

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University of Illinois

Dear Mr. James,

I can attribute some to Mathematics Magazine I'm sure – my son changed his course at Brown University from Engineering to Applied Mathematics. He has just completed his first year so just "laying around the house, (your books, I mean, of course!), has helped at least one student towards Mathematics.

Sincerely

OPERATOR MATHEMATICS III

Jerome Hines

9. The aims of this paper include the more general aims of the previous two of this series. Right and left hand power operators will be defined which can be identified with exponents in the case of linear operators. This leads us, in the case of linear operators, to extend our definitions of exponents. We will define right and left hand exponents which will have the same meaning if, and only if, certain conditions of commutativity are fulfilled. Several properties of the power operators are demonstrated as well as numerous examples. Interesting among these are the formulae used in the classical approach to operators but which previously lacked a solid foundation based upon a calculus of operators such as has been developed in the previous two papers.

10. The Power Operator:

The formula

$$a^b \equiv {}_hL_0 \, [\, 1 + b(a^h - 1)]^{1/h}$$

may be easily verified. We extend the concept of the power operator to include right and left hand subscripts by

10.1 $$_BP \cdot A \overset{S}{=} {}_hL_0 \, [\, I + B(A^h - I)]^{1/h}$$

and

10.2 $$P_B \cdot A \overset{S}{=} {}_hL_0 \, [\, I + (A^h - I)B]^{1/h}$$

10.1 and 10.2 are distinctly different if B and A do not commute, otherwise they are equivalent. Note that the operators $_BP$ and P_B only have meaning when applied to an operator.

Let us investigate $P_B \cdot AP_C \cdot A$:

Since

$$P_B \cdot AP_C \cdot A \overset{S}{=} {}_hL_0 \, [\, I + (A^h - I)B]^{1/h} \; {}_mL_0 \, [\, I + (A^m - I)C]^{1/m}$$

then, letting $m = h$,

$$_BP \cdot AP_C \cdot A \overset{S}{=} {}_hL_0 \, [\, I + (A^h - I)B]^{1/h} \, [\, I + (A^h - I)C]^{1/h}$$

If A, B, and C are linear and commute with each other, then

$$_{B}P \cdot AP_{C} \cdot A \overset{S}{=} {}_{h}L_{0}\{[I + (A^{h}-I)B]\,[I + (A^{h}-I)C]\}^{1/h}$$

$$\overset{S}{=} {}_{h}L_{0}\,[I + (A^{h}-I)B + (A^{h}-I)C + (A^{h}-I)B\,(A^{h}-I)C]^{1/h}$$

We can drop the last term on the right hand side since it is an infinitesimal of higher order than the second term. Then

$$_{B}P \cdot AP_{C} \cdot A \overset{S}{=} {}_{h}L_{0}\,[I + (A^{h}-I)B + (A^{h}-I)C]^{1/h}$$

$$\overset{S}{=} {}_{h}L_{0}\,[I + (A^{h}-I)(B+C)]^{1/h}$$

or

10.3 $$_{B}P \cdot AP_{C} \cdot A \overset{S}{=} P_{B+C} \cdot A$$

for A, B, and C linear and mutually commutative. In the above case $P_B \cdot A \overset{S}{=} {}_{B}P \cdot A$ since A and B commute.

We may further generalize the power operator by the definition:

10.4 $$_{B}P_{C} \overset{S}{=} {}_{h}L_{0}\,[I + B(A^{h}-I)C]^{1/h}$$

where

10.5 a) $_{I}P_{C} \cdot A \overset{S}{=} P_{C} \cdot A$ b) $_{B}P_{I} \cdot A \overset{S}{=} {}_{B}P \cdot A$

10.6 a) $_{\Omega}P_{C} \cdot A \overset{S}{=} I$ b) $_{B}P_{\Omega} \cdot A \overset{S}{=} I$

Let us now investigate $\lg_e \cdot ({}_{B}P_{C} : A)$ where A, B, and C are all linear Note that in the above, according to our conventions in the first paper of this series, we apply $_{B}P_{C}$ to A, then we take the logarithm of this operator and finally apply the resulting operator to whatever operand is furnished. By

7.2:

$$\lg_e \cdot {}_{B}P_{C} : A \overset{S}{=} {}_{k}L_{0}\,\partial_{k}\,\{{}_{h}L_{0}[I + B(A^{h}-I)C]^{1/h}\}^{k}$$

$$\overset{S}{=} {}_{k}L_{0}\,\partial_{k}\,\{{}_{h}L_{0}[I + B(A^{h}-I)C]^{k/h}\}$$

$$\overset{S}{=} {}_{k}L_{0}\,\partial_{k}\,{}_{h}L_{0}\left[I + \frac{k}{h}B(A^{h}-I)C + \frac{k}{h}\left(\frac{k}{h}-1\right)\frac{[B(A^{h}-I)C]^{2}}{2!} + \cdots\right]$$

provided the binomial expansion of this term when applied to the operand is convergent. Then

$$\lg_e \cdot {}_{B}P_{C} : A \overset{S}{=} {}_{k}L_{0}\,\partial_{k}\left[I + kB\lg_e \cdot AC + \frac{k^{2}(B\lg_e \cdot AC)^{2}}{2!} + \cdots\right]$$

$$\lg_e \cdot {}_BP_C : A \overset{S}{=} {}_kL_0 \, [B \lg_e \cdot A C + k (B \lg_e \cdot A C)^2 + \cdots]$$

and

10.7 $$\lg_e \cdot {}_BP_C : A \overset{S}{=} B \lg_e \cdot A C$$

for A, B, and C linear.

11. Exponents and Exponential Forms:

In the first paper of this series we defined an additive index law for all operators so that the exponent of an operator would indicate the number of applications of the operator when the exponent was an integer, i.e.

$$E^p E^q \overset{S}{=} E^{p+q}$$

so that

$$\underbrace{E E E \cdots E}_{n \text{ times}} \overset{S}{=} E^n$$

From the definition of the power operator we can set

11.1 $$P_B \cdot A \overset{S}{=} A^B$$

and

11.2 $$_BP \cdot A \overset{S}{=} {}^BA$$

if, and only if, A and B commute. If B is a constant multiplier, b, then

$$^bA = A^b$$

and the additive index law holds because the conditions of 10.3 are fulfilled. Thus we may utilize 11.1 and 11.2 without contradicting the additive index law if A and B are linear. If they are not linear then 11.1 and 11.2 usually do not hold. If we wished to define our system such that 11.1 and 11.2 held for all operators then we would have to abandon the additive index law for most non-linear operators. In this treatise we shall retain the additive index law and treat $P_B \cdot A$ and A^B as different entities in the case of non-linear operators. In fact we have no definition as yet for A^B if A is non-linear and B is not a constant.

That the additive index law will be used does not imply that

$$A^B A^C \overset{S}{=} A^{B+C}$$

In fact this equation is only true if A, B, and C all commute (A is linear).

12. Applications of Operator Exponents:

We shall derive the equation

$$\delta_x^{a} \overset{S}{=} e^{aD}$$

commonly used in Theory of Differences* but which has been regarded as a mere formal statement of analogy to function theory since it has lacked the background developed in this series of papers. Since

$$e^{aD} \overset{S}{=} {}_hL_0 \; [1 + (e^h - 1)aD]^{1/h}$$

$$\overset{S}{=} {}_hL_0 \left[1 + \frac{1}{h} (e^h - 1)aD + \frac{1}{h} (\frac{1}{h} - 1) \frac{(e^h - 1)^2 a^2 D^2}{2!} + \cdots \right]$$

or

12.1
$$e^{aD} \overset{S}{=} \sum_{i=0}^{\infty} \frac{a^i D^i}{i!}$$

by a binomial expansion as used in proving 11.10. Furthermore

$$\partial_a \cdot \delta_x^a \overset{S}{=} \partial_a \delta_x^a - \delta_x^a \partial_a$$

by the definition of the derivative of a linear operator, where "a" is the independent variable, x held constant. Then

$$\partial_a \cdot \delta_x^a f(x) \equiv \partial_a \delta_x^a f(x) - \delta_x^a \partial_a f(x)$$

But

$$\delta_x^a f(x) \equiv f(x + a)$$

and

$$\partial_a f(x) \equiv 0$$

whence

$$\partial_a \cdot \delta_x^a f(x) \equiv \partial_a f(x + a) \equiv f'(x + a) \equiv \delta_x^a f'(x) \equiv \delta_x^a D_x f(x)$$

or

12.2
$$\partial_a \cdot \delta_x^a \overset{S}{=} \delta_x^a D_x$$

Then

$$\partial_a^2 \cdot \delta_x^a \overset{S}{=} \partial_a \cdot (\delta_x^a D_x) \overset{S}{=} \partial_a \cdot \delta_x^a D_x$$

since

$$\partial_a \cdot D_x \overset{S}{=} \Omega \quad \text{and} \quad \partial_a^2 \cdot \delta_x^a \overset{S}{=} \delta_x^a D_x^2$$

Similarly

12.3
$$\partial_a^n \cdot \delta_x^a \overset{S}{=} \delta_x^a D_x^n$$

*See: Boole, Theory of Finite Differences; Stecher Reprint, 1926, pg. 18.
Since ${}_a\delta_x \overset{S}{=} {}_1\delta_x^a$ we will also denote ${}_a\delta_x$ by δ_x^a.

and

$$ {}_0\partial_a^n \cdot \delta_x^a \overset{S}{=} \delta_x^0 D_x^n \quad \text{where} \quad {}_0\partial_a^n f(a) \equiv f^{(n)}(0) $$

But

$$ \delta_x^0 f(x) \equiv f(x+0) \quad \text{or} \quad \delta_x^0 \overset{S}{=} I $$

whence

12.4 $$ {}_0\partial_a^n \cdot \delta_x^a \overset{S}{=} D_x^n $$

By section 4 of the first paper of this series it was shown that an arbitrary operator having an infinite number of derivatives can be expressed by

$$ A \overset{S}{=} \sum_{i=0}^{\infty} \frac{(a-a_0)^i \; {}_{a_0}D_a^i \cdot A}{i!} $$

Let $A = \delta_x^a$, and $a_0 = 0$, then

12.5 $$ \delta_x^a \overset{S}{=} \sum_{i=0}^{\infty} \frac{a^i D_x^i}{i!} $$

We could have seen this easily by the Taylor expansion of $f(x+a)$ but the above developement is quite instructive.

Since the right hand side of 12.5 is the same as the right hand side of equation 12.1 we have proven that

12.6 $$ \delta_x^a \overset{S}{=} e^{aD} $$

The more general case of $e^{g(x)D}$ is easily found:

$$ e^{g(x)D_x} f(x) \equiv e^{D_{P(x)}} f(x) \quad \text{where} \quad P(x) \equiv \int \frac{dx}{g(x)} $$

If a unique inverse exists for $P(x)$ then we may write

$$ f(x) \equiv f\{P^{-1}[P(x)]\} $$

and

$$ e^{g(x)D_x} f(x) \equiv e^{D_{P(x)}} f(x) \equiv e^{D_{P(x)}} f\{P^{-1}[P(x)]\} $$

Since

$$ \delta_{P(x)}^1 G[P(x)] \equiv e^{D_{P(x)}} G[P(x)] \equiv G[P(x)+1] $$

Then

12.7 $$ e^{g(x)D} f(x) \equiv f\{P^{-1}[P(x)+1]\} $$

where

$$ P(x) \equiv \int \frac{dx}{g(x)} $$

This formula can be found in the classical literature* and now has a rigorous foundation.

12.7 has many interesting applications. For example:

$$a^{xD} \overset{S}{=} e^{x \lg_e \cdot aD}$$

whence

$$P(x) \equiv \frac{1}{\lg_e \cdot a} \int \frac{dx}{x} \equiv \frac{\lg_e \cdot x}{\lg_e \cdot a}$$

$$P^{-1}(x) \equiv e^{x \lg_e \cdot a}$$

and

$$P^{-1}[P(x) + 1] \equiv e^{\lg_e \cdot a\left(\frac{\lg_e \cdot x}{\lg_e \cdot a} + 1\right)} \equiv e^{\lg_e \cdot a \frac{\lg_e \cdot (ax)}{\lg_e \cdot a}} \equiv ax$$

whence

12.8 $$a^{xD} f(x) \equiv f(ax)$$

If we define the multiplication operator, ${}_a\mu_x$, by

$${}_a\mu_x \, f(x) \equiv f(ax)$$

then by 12.8,

12.9 $${}_a\mu_x \overset{S}{=} a^{xD}$$

This formula is also well known** but has lacked a proper developmental background.

If we define the negation operator, N_x, by

$$N_x f(x) \equiv f(-x)$$

then by 12.9,

12.10 $$N_x \overset{S}{=} (-1)^{xD}$$

Obviously

$$N_x^2 \overset{S}{=} I$$

This can easily be verified by using $(-1)^{xD}$

Let us determine $q(x)$ such that it will satisfy the equation

$$e^{q(x) D_x} f(x) \equiv f(x^r)$$

Then, by 12.7

$$e^{D_{P(x)}} f(x) \equiv f\{P^{-1}[P(x) + 1]\} \equiv f(x^r)$$

*C. Graves: A Generalization of the Symbolic Statement of Taylor's Theorem. Proc. Royal Irish Academy, Vol. 5 (1850-1853), pp. 285-287.

**A A AAA

2nd Reference: The Theory of Linear Operators, by Harold T. Davis, pg. 13.

or

$$x^r \equiv P^{-1}[P(x)+1]$$

whence

$$P(x+1) \equiv P(x^r)$$

But this equation is satisfied by

$$P(x) \equiv \lg_r \lg_m \cdot x \equiv \frac{1}{\lg_m \cdot r} \lg_m^2 \cdot x$$

by 11.9. For convenience we shall choose $m = e$, i.e.

$$P(x) \equiv \frac{1}{\lg_e \cdot r} \lg_e^2 \cdot x$$

But

$$q(x) \equiv \frac{1}{P'(x)}$$

Since

$$P'(x) \equiv \frac{1}{\lg_e \cdot r} D \lg_e^2 \cdot x \equiv \frac{1}{\lg_e \cdot r} \frac{1}{x \lg_e \cdot x}$$

and

$$\frac{1}{P'(x)} \equiv \lg_e \cdot r \, x \lg_e \cdot x$$

then

$$e^{q(x)D} \overset{S}{=} [x^{x \lg_e \cdot r}]^D \overset{S}{=} [r^{x \lg_e \cdot x}]^D$$

i.e.

12.11 $$[x^x]^{\lg_e \cdot rD} f(x) \equiv f(x^r)$$

For $r = e$

12.12 $$(x^x)^D f(x) \overset{S}{=} f(x^e)$$

For $r = -1$, where $\lg_e \cdot (-1) = n\pi i$, for n an integer, then

12.13 $$x^{n\pi i xD} f(x) \equiv f(\frac{1}{x})$$

If we define the invert operator, $\ln_x$, by

$$\ln_x f(x) \equiv f(\frac{1}{x})$$

then

12.14 $$\ln_x \overset{S}{=} x^{n\pi i xD}$$

It is interesting to investigate an operator like D^D:

$$D^D \overset{S}{=} {}_hL_0\,[I + (D^h - I)D]^{1/h}$$

12.15 $$D^D \overset{S}{=} \sum_{i=0}^{\infty} \frac{D^i(\lg_e \cdot D)^i}{i!}$$

using a binomial expansion and taking the limit.

If we apply this to e^{ax} using 8.2 from the second paper of this series, i.e.,

8.2 $$\lg_e \cdot D e^{ax} \equiv e^{ax} \lg_e \cdot a$$

then

$$(D \lg_e \cdot D)^i \; e^{ax} \equiv (\lg_e \cdot a^a)^i \, e^{ax}$$

and

$$D^D e^{ax} \equiv e^{ax} \sum_{i=0}^{\infty} \frac{(\lg_e \cdot a^a)^i}{i!}$$

whence

12.16 $$D^D e^{ax} \overset{S}{=} a^a e^{ax}$$

By 8.3 of the second paper

$$\lg_e \cdot D \sin x \equiv \frac{n\pi}{2} \cos x \qquad (n \text{ any integer})$$

and

$$D \lg_e \cdot D \sin x \equiv -\frac{n\pi}{2} \sin x$$

$$(D \lg_e \cdot D)^2 \sin x \equiv \left(\frac{n\pi}{2}\right)^2 \sin x$$

and

$$(D \lg_e \cdot D)^i \sin x \equiv \left(-\frac{n\pi}{2}\right)^i \sin x$$

whence

$$D^D \sin x \equiv \sum_{i=0}^{\infty} \frac{(D \lg_e \cdot D)^i \sin x}{i!} \equiv \sum_{i=0}^{\infty} \frac{\left(-\frac{n\pi}{2}\right)^i}{i!} \sin x$$

and

12.17 $$D^D \sin x \equiv e^{-\frac{n\pi}{2}} \sin x$$

Similarly

12.18 $$D^D \cos x \equiv e^{-\frac{n\pi}{2}} \cos x$$

Also $D^{iD} \sin x \equiv -i \sin x$ and $D^{iD} \cos x \equiv -i \cos x$.

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher,* are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Aflred, New York.

The Fifth Annual
WILLIAM B. ORANGE

MATHEMATICS PRIZE COMPETITION

for Students in High Schools of the Los Angeles School District

(Editor's Note: While this report is local, it probably is fairly representative of similar groups of high school students in the United States. We would welcome reports on similar tests from any or all of the other 45 countries into which the Mathematics Magazine goes. We give so much space to these questions because they seem extremely interesting per se.)

Part I (50%) ONE HOUR

Check correct answer on answer sheet.

1. A boy takes 30 minutes to go one mile to school, but gets home in 7½ minutes. His average speed in miles per hour for the round trip is:

   (a) 5 (b) 4 (c) 3.2 (d) 3.125 (e) 3

2. At what rate should a merchant discount an item already discounted at 20% so as to give a total 40% discount?

   (a) 10% (b) 15% (c) 20% (d) 25% (e) 30%

3. $\left[a^{\frac{1}{n+1}} \cdot a^{\frac{1}{n-1}} \right]^{1-n^2}$ equals:

   (a) a^{-1} (b) a^{-2n} (c) $-a^{-2n}$ (d) $-a$ (e) a^{-2}

4. If r_1 and r_2 are the roots of the equation $ax^2 + bx + c = 0$, select from the following the equation whose roots are r_1^2 and r_2^2:

(a) $a^2x^2 + b^2x + c^2 = 0$
(b) $a^2x^4 + b^2x^2 + c^2 = 0$
(c) $(ax^2 + bx + c)^2 = 0$
(d) $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$
(e) $a^2x^2 - (b^2 - 4ac)x + c^2 = 0$

5. A punch board containing 50 chances ranging in cost from one to to fifty cents without duplication gives a prize of \$5.00. Assuming honesty in handling, the operator's profit is:

(a) \$5.00 (b) \$7.50 (c) \$7.75 (d) \$10.00 (e) nothing

6. Three coins of diameter $2a$ are placed flat on a table top. What is the diameter of the smallest circle that can enclose them?

(a) $4a$ (b) $\frac{4a\sqrt{3} + 6a}{3}$ (c) $\frac{8a\sqrt{3}}{3}$ (d) $4a\sqrt{3} - 3a$ (e) $3a\sqrt{2}$

7. Which of the following is irrational?

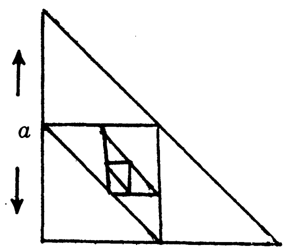
(a) 0.3153153153153153... (b) 0.365278

(c) the infinite decimal $0.a_1a_2a_3 \ldots$ where $a_{2k} = 3$ and $a_{2k-1} = 2$ for all positive integral values of k

(d) the solution of $x = 1/(1 + x)$

(e) $\sqrt{b^2 - 4ac}$ if $ax^2 + bx + c = (px + q)(rx + s)$ where a, b, c, p, q, r, s are integers

8. The figure consists of isosceles right triangles drawn as shown and the process continued indefinitely. The total length of all line segments in the figure is:



(a) $4a + a\sqrt{2}$
(b) $5a\sqrt{2}$
(c) $2a(2 + \sqrt{2})$
(d) $8a$
(e) infinite

9. The ratio of the area of a square inscribed in a circle to the area remaining when the square is removed is:

(a) 2 to 1 (b) 2 to π-2 (c) π to 2 (d) π-1 to 2 (e) depends on the radius of the circle.

10. If m and n are positive integers greater than 1, which of the

following is the largest?

(a) $m + n$ (b) $\sqrt{2mn}$ (c) $\dfrac{m^2 + n^2}{m + n}$ (d) $\dfrac{m^3 + n^3}{m^2 + n^2}$ (e) More information needed

11. A set of points in a plane is defined to be convex if the line segment joining any two points in the set is completely contained in the set. Which of the following is not convex?

(a) all the points on and in a circle
(b) all the points lying between two parallel lines
(c) all the points on a straight line
(d) same as (a) except one point on the boundary removed
(e) same as (c) except one point is removed

12. The largest number by which the expression

$$(n+5)(n+6)(n+7)(n+8)(n+9)$$

is divisible for all possible integral values of n is:

(a) 60 (b) 120 (c) 360 (d) 720 (e) 30

13. $\log_2 x = (\log_x 2)^{-1}$ is true for what positive values of x?

(a) 2 only (b) 2 and 4 only (c) integral powers of 2 only
(d) all positive values of x (e) no positive values of x

14. The mistake in the following occurs between which two steps?

1. let $a = b$
2. $a^2 = ab$
3. $a^2 - b^2 = ab - b^2$
4. $(a + b)(a - b) = b(a - b)$
5. $a + b = b$
6. $2b = b$
7. $2 = 1$

(a) 1 and 2 (b) 2 and 3 (c) 3 and 4 (d) 4 and 5 (e) 5 and 6

15. Perpendiculars are drawn to the 3 sides of an equilateral triangle of side s from any point inside the triangle. Which of the following is *not* true?

(a) The 3 quadrilaterals formed are each inscribable in a circle
(b) The angles of the 3 quadrilaterals are respectively equal to each other
(c) The sum of the 3 perpendiculars equals the altitude
(d) The sum of the perimeters of the 3 quadrilaterals equals twice the perimeter of the triangle
(e) The total area of the 3 quadrilaterals equals $\frac{s^2}{4}\sqrt{3}$

16. One penny is rolled completely around another, both flat on the table. How many revolutions about its own center does the moving coin make?

(a) ½ (b) 1 (c) 2 (d) 4 (e) none

17. The second and fifth terms of a geometric progression are 2 and 6 respectively. The first term is:

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt[3]{3}$ (d) $\frac{2}{\sqrt[3]{3}}$ (e) $\frac{\sqrt{3}}{3}$

18. A cuble four inches on an edge is painted blue and then cut into one inch cubes. How many of the small cubes have exactly 2 surfaces blue?

(a) 16 (b) 24 (c) 27 (d) 32 (e) 64

19. At what time do the minute and hour hands of a clock form an angle of 105°?

(a) 12:20 (b) 1:25 (c) 2:30 (d) 3:35 (e) all of the time

20. If the base of our number system were 9, the product of 27 by 35 in that system would be:

(a) 1078 (b) 978 (c) 945 (d) 1045 (e) 1000

21. The altitude of a triangle inside the triangle is the arithmetic mean between the two unequal segments of the base. What kind of a triangle is it?

(a) right (b) acute (c) obtuse (d) isosceles (e) need more information

22. The cost of a/b automobiles was b/c dollars. The selling price was c/a dollars each. The profit was what fraction of the selling price?

(a) $1 - \frac{a^2}{c^2}$ (b) $1 - \frac{b^2}{c^2}$ (c) $\frac{ab}{c} - 1$ (d) $\frac{a}{c}$ (e) $\frac{a^2 - c^2}{ac}$

23. The perimeter of a rectangle is 80 and its area is A. Its diagonal is:

(a) $2\sqrt{400 - A}$ (b) $20\sqrt{2}$ (c) $\sqrt{1600 - 2A}$ (d) $10\sqrt{10}$
(e) none of these

24. A square sheet of metal of side length a is to have equal small squares cut from the corners, and is then to be folded into a box having a volume numerically equal to k times the side of the square. The side of the small square is:

(a) $a - 2k$ (b) $a - k$ (c) $\sqrt{a - k}$ (d) $(\sqrt{a} - k)/2$ (e) $(a - \sqrt{k})/2$

25. Five books can be stacked in how many different ways?
(a) 25 (b) 32 (c) 24 (d) 60 (e) 120

Part II (50%) 1½ HOURS

1. Let P_n represent the number of parts into which the plane is divided by n straight lines, no two of which are parallel and no no three of which are concurrent.

   (a) Find and prove a formula for P_{n+1} in terms of P_n and n; i.e., $P_{n+1} = P_n + ?$

   (b) Find and prove a formula for P_n in terms of n.

2. In a recent issue of *Scripta Mathematica,* there appeared an item by C. W. Trigg entitle "Playing with the Didgits of 1954." The item included the following equalities:

$$1 + 9 + 5 + 4 = 4 + 5 + 9 + 1$$

$$19 + 95 + 54 + 41 = 14 + 45 + 59 + 91$$

   (a) Write a similar equality using the digits 3 at a time.
   (b) Given a number of n digits represented by $a_1 a_2 a_3 \ldots a_n$ (For example, in the case of 1954, $n = 4$, and $a_1 = 1$, $a_2 = 9$, $a_3 = 5$, $a_4 = 4$)

   The following equality states the property indicated in (a):

$$a_1a_2a_3 + a_2a_3a_4 \ldots + \_\_ + \_\_ = \_\_ + \_\_ + \ldots a_4a_3a_2 + a_3a_2a_1$$

   Complete the blanks.

   (c) Prove the property expressed in (b).

   (d) Generalize the problem above in any way you can. Try to prove your conjectures.

* * * * * * * * *

SCORE	Number of Contestants Part I	Part II	Average
90 - 100	0	0	0
80 - 89	4	3	2
70 - 79	4	2	2
60 - 69	14	5	6
50 - 59	19	8	9
40 - 49	48	12	24
30 - 39	34	19	38
20 - 29	33	32	46
10 - 19	14	51	38
0 - 10	1	39	6
	171	171	171

Average, Part I : 40
Average, Part II : 24
Average, Combined: 32

Part I Problem Number	Number of Correct Answers (171 contestants)	Part I Problem Number (continued)	Number of Correct Answers (continued)
2	136	23	64
5	124	15	63
3	120	11	62
10	113	6	49
17	102	22	47
18	100	24	46
14	97	13	40
25	78	19	39
1	77	20	31
9	77	21	31
7	71	4	30
8	64	12	25
		16	24

AN APPEAL FOR MORE SCIENCE STUDENTS!

An appeal to the Nation's publishers to help stimulate interest in high-school mathematics and science courses has been made by the president of the National Society of Professional Engineers. In a letter addressed to approximately 800 publishers of daily newspapers, Allison C. Neff, of Middletown, Ohio, newly installed president of the 36,000-member engineering group, asked the help of the press as a first step to "stem the tide of students away from elementary mathematics, physics and chemistry."

Neff, vice president of Armco Drainage and Metal Products, Inc., asked publishers to consider these facts: (i) only about 25 percent of our high-school students now study algebra; (ii) only about 12 percent of our high-school students now study geometry; (iii) there are many small, and even some large, high schools that offer no courses in the physical sciences.

From *SCIENCE*, Aug. 19, 1955

You can help by writing for our "Semi-popular and Popular Department." Editor.

COMPLEX QUANTITIES IN THE FIRST COURSE IN DIFFERENTIAL EQUATIONS

HUGH. J.. HAMILTON

1. An extract from the Standard Junior-year Course. The general linear, ordinary differential equation with constant (real) coefficients,

$$\sum_{k=0}^{n} a_k d^k y/dx^k = f(x) \qquad (a_n = 1),$$

is customarily written as

(1) $$F(D)y = f(x),$$

where D represents the operator d/dx, $F(D) = \sum_{k=0}^{n} a_k D^k$, and the algebra involved is self-explanatory. The general solution of (1) may be written, with further symbolism, as

(2) $$y = \frac{1}{F(D)} f(x).$$

In particular

(3) $$\frac{1}{D^n} 0 = \sum_{k=1}^{n} c_k x^{k-1}$$

where the c_k are the arbitrary constants. The linearity of the operator $F(D)$ implies the fundamental relation

(4) $$\frac{1}{F(D)} f(x) = \frac{1}{F(D)} 0 + \left[\frac{1}{F(D)} f(x)\right]_0$$

where the subscript signifies that any *particular integral* taken from (2) may be used as the expression in square brackets; the term $\frac{1}{F(D)} 0$, which involves the n arbitrary constants of the general solution of (1), is the "complementary function" and is the general solution of the "homogeneous equation corresponding to (1)," namely

(5) $$F(D)y = 0.$$

Hardly less fundamental then (4) is the "Translation Theorem,"

(6) $$\frac{1}{F(D)} e^{ax}\phi(x) = e^{ax}\frac{1}{F(D+a)}\phi(x),$$

which is consequence of the identity in $g(x)$:

$$(d+a)^k e^{-ax} g(x) = e^{-ax} D^k g(x)$$

(for then

$$F(D)+a)e^{-ax}\frac{1}{F(D)}e^{ax}\phi(x)=e^{-ax}F(D)\frac{1}{F(D)}\ e^{ax}\phi(x)=\phi(x)\ ).$$

Here a is assumed to be real.

If the auxiliary equation in m,

(7) $$\sum_{k=0}^{n} a_k m^k \equiv 0.$$

has only real roots, then it is easy to show that

(8) $$F(D)=\pi j=1(D-m_j)^{n_j},$$

where the m_j are these roots (all distinct), the n_j are their respective multiplicities, and the order of the factors is immaterial. It is now clear that any solution of

(9) $$(D-m_j)^{n_j} y = 0$$

for any j is also a solution of (5). But, by (6) and (3),

$$\frac{1}{(D-m_j)^{n_j}}0 = \frac{1}{(D-m_j)^{n_j}}\ e^{m_j x}o = e^{m_j x}\ \frac{1}{D^{n_j}}0 = e^{m_j x}\sum_{k=1}^{n_j} c_{jk}x^{k-1}$$

Therefore $y = e^{m_j x}\sum_{k=1}^{n_j} c_{jk}x^{k-1}$ is a solution of (5); since, moreover, the n coefficients $e^{m_j x}x^{k-1}$ of the c_{jk} are linearly independent, this is the *general solution* of (5) and hence the complementary function.

Various devices may be used to find particular integrals of equations (1) for insertion into the expansion (4), the Translation Theorem (6) being frequently of considerable help.

2. The Fly in the Ointment. Most of us who teach differential equations on the junior level present some such symbolic development as that outlined in **1**; it is straightforward and, if taught leisurely and with numerous examples, presumably understood by the student. There would seem to be no question as to its pedagogical soundness.

There *is* a question as to the appropriateness of using (8) and (9) when some of the roots of (7) are complex, and of using (6) when a is complex - as when, wishing to simplify

(10) $$\frac{1}{F(D)}\cos bx\ \phi(x)\quad\text{or}\quad\frac{1}{F(D)}\sin bx\ \phi(x),$$

we begin by writing the circular function in exponential form. There are serious problems, first of meaning, then of validity, and we do not have time to give a subsidiary course in complex variable at this point. In favor of using the complex forms freely and with little apology are these arguments: that the solution of many problems is thereby very greatly facilitated, that even applied mathematicians who have never heard of function theory use complex variables without qualms and without error, and that the student cannot be too often impressed with the persistence of form in mathematical truths. The main argument against using the complex forms is that students are already too gullible ("If $D^2 - a^2 = (D + a)(D - a)$, then any fool can see that $D^2 + b^2 = (D + ib)(D - ib)$"); to encourage them in this is to teach science badly and to betray one of the cleanest educational purposes of mathematics; the stimulation of critical thinking.

Ideally, perhaps, we would so teach our students that they felt privileged to use the complex forms at all times, but only with a tender conscience and always with impatient anticipation of the day when they would be able to construct their own proofs of validity. Practically, of course, we may congratulate ourselves if we can get them to agree that anything *needs* to be proved.

3. A Possible Way Out. Between the logically unsound approach whose first step is to pretend that $De^{ix} = ie^{ix}$ "simply because i is a constant" and the psychologically unsound approach which lumbers along with real-variable analogs of (6), such as

$$\frac{1}{D^2 + c^2} \cos bx\, \phi(x) =$$

$$\left\{\cos bx\, [D^2 + (c^2 - b^2)] + \sin bx\, [2bD]\right\} \frac{1}{D^2 + (c^2 - b^2)^2 + 4b^2D^2} \phi(x),$$

lies a possibly pedagogically sound approach by way of some rudimentary caluclus of a complex function of a real variable. The remainder of this section shows how such an approach can be made.

Familiarity with informal complex algebra presupposed, we consider $y \equiv u + iv$, where u and v are n-times differentiable functions of x. We define $(D - a - ib)y \equiv Dy - (a + ib)y$, and we define $Dy \equiv Du + iDv$ (both definitions being in fact dictated, if we require linearity of our operations). It is now easy to show that the general differentiation formulas of elementary calculus carry over to complex functions of x, that $F(D)$ is linear with respect to such functions, and that everything stated in para. 1. which does not involve complex exponentials is meaningful and valid whether or not the a_k in (1) or the roots of (7) are real.

Turning next to the complex exponential, we recall that defining properties of $y \equiv e^{ax}$ for real a are

$$(11) \qquad Dy = ay, \quad y = 1 \text{ when } x = 0$$

We are thus led to define

$$(12) \qquad z \equiv e^{(a+ib)x}$$

by the conditions

$$(13) \qquad Dz = (a + ib)z,$$

$$(14) \qquad z = 1 \text{ when } x = 0.$$

Putting

$$(15) \qquad z = u + iv$$

in (13), we obtain the simultaneous system.

$$(16a) \qquad (D - a)u + bv = 0,$$

$$(16b) \qquad bu - (D - a)v = 0.$$

Operation on (16a) by $(D - a)$ and multiplication of (16b) by b, followed by addition of the results, yields the equation $[(D - a)^2 + b^2]u = 0$, whence, by (6), we obtain

$$(17) \quad u = \frac{1}{(D-a)^2 + B^2}\,0 = \frac{1}{(D-a)^2+b^2}\,e^{ax}0 = e^{ax}\,\frac{1}{D^2+B^2}\,0 \equiv e^{ax}w,$$

where

$$(18) \qquad w'' + b^2 w = 0.$$

If we multiply (18) by the integrating factor $2u'$ and integrate twice, we obtain $w = C\cos(bx + c)$, so that, by (17), we have $u = Ce^{ax}\cos(bx + c)$. Substituting this expression for u into (16a), we find that $v = Ce^{ax}\sin(bx + c)$ and thus, by (15), that $z = Ce^{ax}[\cos(bx + c) + i\sin(bx + c)]$. Condition (14) now requires that $1 = C(\cos c + i \sin c)$, whence $C = 1$ and c is an even multiple of π. Therefore, finally, by (12), we have

$$(19) \qquad e^{(a+ib)x} \equiv e^{ex}(\cos bx + i \sin bx).$$

Since it was only the formal fact of (11) that validated (6) in the first place, the holding of condition (13) now enables us to assert the validity of (6) for complex a, under the definition (19). Hence everything stated in para. **1.**, without exception, is meaningful and valid, whether or not the a_k in (1) or the roots of (7) are real.

There remain problems of interpretation. From (19), by first taking $a = 0$, and then taking $a = 0$ and at the same time replacing b by $-b$, we get two equations which yield the exponential formulas for cos bx and sin bx; these can be used, with (6), to simplify expressions like those in (10). If the a_k in (1) are real, then we deduce from (7) and (8) that to each quation (9) with non-real $m_j \equiv a + ib$ there

(20) $$(D - \bar{m}_j)^{n_j} y = 0$$

where $\bar{m}_j = a - ib$; and the terms in the complementary function corresponding to (9) and (20) can be grouped thus:

$$e^{m_j x} \sum_{k=1}^{n_j} c_{jk} x^{k-1} + e^{\bar{m}_j x} \sum_{k=1}^{n_j} b_{jk} x^{k-1} =$$

$$e^{ax} \sum_{k=1}^{n_j} (c_{jk} e^{ibx} + b_{jk} e^{-ibx}) x^{k-1}$$

$$e^{ax} \sum_{k=1}^{n_j} [(c_{jk} + b_{jk}) \cos bx + i(c_{jk} - b_{jk}) \sin bx] x^{k-1} =$$

$$e^{ax} [\cos bx \sum_{k=1}^{n_j} c'_{jk} x^{k-1} + \sin bx \sum_{k=1}^{n_j} b'_{jk} x^{k-1}],$$

where the c'_{jk} and the b'_{jk} may evidently be taken independently arbitrary, and *real*.

4. Conclusion. In a sense, the development in para. 3. is a compromise. It does little justice to the unified theory of complex variable; it does not even explain that, at the point where we wrote $y = u + iv$, we had to reinterpret all of para. 1. in terms of its "complex-real" isomorph. On the other hand, to write

$$e^{it} \sum_{n=0}^{\infty} (it)n/n! = \sum_{n=0}^{\infty} (-1)^k t^{2k}/(2k)! +$$

$$i \sum_{k+0}^{\infty} (-1)^k t^{2k+1}/(2k+1)! = \cos t + i \sin t$$

is effectively to stun the student by a virtuoso exhibition into forgetting to ask such truly relevant questions as this; "How should I know that the derivative of ix is i?" Indeed, how should a college junior know what is meant by an infinite series of complex numbers?

And yet, the formal series argument for Euler's Formula *might* bo tantalize the student that he would study more mathematics than if he had never seen it ... .

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On A Convergence Test For Alternating Series

R. Lariviere

Many calculus tests give the following test for the convergence of an alternating series without giving any examples of series which do not satisfy both its requirements.

"An alternating series $u_1 - u_2 + u_3 - u_4 + \dots\dots\dots$, in which the u's are all positive, converges if (a) $\lim_{n\to\infty} u = 0$, and if (b) for terms beyond a certain kth term, $u_n \geqq u_{n+1}$, $n = k, k+1, k+2, \dots\dots$"

The beginner often concludes that the alternating series which do not satisfy both (a) and (b) are necessarily divergent. This error may be obviated by showing him examples of both convergent and divergent series that do not satisfy (b), preferably examples that seem to have uniform general terms such as the following:

$$(1)\ S_1 = \sum_1^{\infty} (-1)^{n+1} \frac{|\sin n|}{n^2}, \quad (2)\ S_2 = \sum_1^{\infty} (-1)^{n+1} \frac{(\sqrt{2})^{(-1)^{n+1}}}{n\quad n}$$

In (1) S_1 is obviously convergent but

$$\frac{u_{n+1}}{u_n} = \frac{n^2}{(n+1)^2} \; \frac{|\sin(n+1)|}{|\sin\ n|}, \quad \frac{u_{n+1}}{u_n} = |\cos 1 + \sin 1 \cot n| \cdot \frac{n^2}{(n+1)^2}$$

Hence (b) is not satisfied.

$$\frac{u_{n+1}}{u_n} = \left(\frac{n}{n+1}\right) (\sqrt{2})^{[(-1)^{n+2} - (-1)^{n+1}]} = \left(\frac{n}{n+1}\right) 2^{(-1)^n}.$$

This ratio is of greater inequality when n is even, and of lesser inequality when n is odd. Hence (b) is not satisfied. S_2 may be shown to be divergent as follows:

$$S_2 = \frac{\sqrt{2}}{1} - \frac{\sqrt{2}^{\,-1}}{2} + \frac{\sqrt{2}}{3} - \frac{\sqrt{2}^{\,-1}}{4} + \frac{\sqrt{2}}{5} - \frac{\sqrt{2}^{\,-1}}{6} + \dots\dots\dots$$

Since $\lim_{n\to\infty} u_n = 0$, we may group these terms in pairs, and number them $2n+1$, $2n+2$ in pairs, with $n = 0, 1, 2 \dots\dots\dots$, as follows:

$$S_2 = \left[\frac{\sqrt{2}}{1} - \frac{\sqrt{2}}{4}\right] + \left[\frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{8}\right] + \left[\frac{\sqrt{2}}{5} - \frac{\sqrt{2}}{12}\right] + \dots + \left[\frac{\sqrt{2}}{2n+1} - \frac{2}{4n+4}\right] + \dots\dots$$

$$S_2 = \sum_0^{\infty} \left[\frac{\sqrt{2}}{2n+1} - \frac{\sqrt{2}}{4n+4}\right] = \sum_0^{\infty} \frac{\sqrt{2}}{4} \cdot \frac{1}{n+1} \cdot \frac{2n+3}{2n+1}$$. Therefore S_2 is divergent.

SEMI-POPULAR AND POPULAR PAGES

IRRATIONAL NUMBERS

Louis E. Diamond

Foreword

This article briefly discusses *only those properties* of integers and rational numbers which are *essential* for the *principal discussion,* the concept of irrational numbers.

Integers

When numbers are restricted to integers, there is an endless number of them but there exists a greatest negative integer, -1, and a least positive integer, +1. The word "bound" implies a limiting number that cannot be exceeded. The set of positive integers is unbounded above, i.e. whatever number, n, is given as an upper bound for the positive integers, there always exists a number, $(n + 1)$, greater than n. All the negative integers, zero, and unity are lower bounds for the positive integers. Unity is the *greatest lower bound* for the positive integers. Similarly, the negative integers are unbounded below and -1 is their *least upper bound.*

A set is said to be dense if between any two members of the set, m, and n, m less than n, there always exists a third member, r, such that $m<r<n$. The set of integers is not dense since no integer exists between any to consecutive integers, as, for example, 1 and 2.

The set of integers is not closed under the operation of division, i.e. division of integers by an integer does not always yield an integer. The restriction that division by zero is meaningless doess hold.

The operation of evolution is not always possible. There exists no number of the set which is the even root of a negative number since the even power of both a negative and a positive number is a positive number. Moreover, there exists no integer which multiplied by itself yields the number two. In this respect two is not by any means unique. Three, five, seven, eight, ten, and in fact most other integers have no square roots.

The integers are said to be linearly ordered, that is, they satisfy the following requirements. Let a, b, and c be arbitrary integers.

1. $a \neq b$ implies $a > b$ or $b > a$.
2. $a > b$ implies $a > b$, i.e. $a > a$ is false.
3. $a > b$ and $b > c$ imply $a > c$.

Rational Numbers

Until a rational number is later defined as a section or cut, let us think of it for the present as indicated algebraically by the ratio, a/b, a any integer whatsoever, b any non-zero integer whatsoever. Ordinarily the integer, a, is called the numerator, the non-zero integer, b, the denominator of the fraction, a/b. We repeat that an essential difference between numerator and denominator is that the denominator must be a non-zero integer.

To ensure in the elementary operations logical consistency with the integers, the following rules are adopted. They are illustrated symbolically.

1. *Equality.* By definition $a/b = c/d$ if, and only if, $ad = bc$. The set of rational numbers includes the set of integers as a subset. Merely set $b = 1$. $a = a \cdot 1$. $a/1 = a$.

(Technically a couple of integers, $(a,1)$, or the ratio $a/1$, is not the same thing as an integer, a. However, each integer a determines a couple $(a,1)$ or a ratio $a/1$, and this couple or ratio behaves under addition and multiplication exactly like the integer a itself. The one-to-one correspondence $a \longleftrightarrow (a,1)$ is called an isomorphism of the domain of integers to a subset of the domain of rationals. However, the integers are not uniquely defined by their postulates but only up to an isomorphism and no algebraic properties are lost if each integer a is simply identified with the corresponding ratio, $a/1$.)

The numbers 2/4, 4/8 and 1/2 are all merely different ways of presenting the same rational number. The number 1/2 is considered the simplest because 1 and 2 contain no factors common to both except unity. The number a/b is said to be in its lowest form if a and b are relatively prime, i.e. contain no factor common to both except unity

2. $+a/-b = -a/+b$. Hence a rational number can always be written with a positive denominator.

3. *Inequalities.* Rational numbers a/b and c/d are written with positive denominators. $a/b > c/d$ if, and only if, $ad > bc$. $a/b < c/d$ if and only if $ad < bc$. Let a/b be a positive rational number, $a > 1$.

$$a/b > a/(b+1) \text{ since } ab + a > ab.$$

$$a/(b+1) > 1/(b+1) \text{ since } a(b+1) > 1(b+1)$$

$$\therefore a/b > a/(b+1) > 1/(b+1) \text{ set } n = b+1,$$

$$a/b > a/n > 1/n.$$

Irrespective of how close to zero we choose the number a/b, a number closer to zero can be found by a suitable selection of n. It is meaningless to speak of a least positive rational number. Similarly, by approaching zero through negative rational numbers it can be shown that there is no greatest negative rational number.

4. *Addition.* $a/b \pm c/d = (ad \pm bc)/bd$. Between any two rational numbers, $a/b > c/d$, a third rational number can always be found. One method is the following.

$$\frac{a/b + c/d}{2} = \frac{ad + bc}{2bd}$$

$$a/b > \frac{ad + bc}{2bd} > c/d.$$

Between any two integers, as, for example, 1 and 2, there exists an infinite number of rational numbers. Two rational numbers are either equal or unequal. Between any two unequal rational numbers, no matter how small their numerical difference, there always exists an endless number of unequal rational numbers. Technically, the rational numbers are "dense." Hence, we can speak of the rational numbers as a linearly ordered dense set.

Geometrically there is an infinite number of rational points on any line segment. In making this statement we have no very large number in mind. Choose two rational numbers whose numerical difference is as small as you desire. Irrespective of whether we use the words, one million, one billion, or one trillion, *we are no nearer to expressing the totality of rational numbers between them than if we had used the word one.*

The symbol ∞ is not a number. It is merely a shorthand expression for the concept of endless numbers. $\lim_{n\to\infty} 1/n = 0$. This expresses symbolically what was previously shown. As the integer n assumes constantly and endlessly increasing values, the rational number, $1/n$, approaches, as a limiting value, zero, although it never reaches that value.

5. *Multiplication.* $(a/b)(c/d) = ac/bd$. In particular, if $a \neq 0, b \neq 0$, $(a/b)(b/a) = ab/ba = ab/ab = 1$. Two rational numbers whose product is unity are called inverse elements under multiplication, or reciprocals. Every rational number except zero has a unique inverse element.

$bx = a$, $b \neq 0$, is always uniquely solvable. $(1/b)(b)x = (1/b)a$. $x = a/b$. We are assured that the number $1/b$ exists and hence we can multiply by it. Hence, except for division by zero, the set of rational numbers is closed under the operation of division.

The Rational Numbers as Sections or Cuts

The manner in which we now define rational numbers will seem farfetched and quite artificial. But, as will appear shortly, there is a valid reason for this definition, prior to a consideration of irrational numbers. For the following definition to be valid, the set must be linearly ordered and dense. To be specific, the rational number 1/3, will be defined. Four sequences are listed below. In each sequence the number of members is endless. *a.* and *b.* are written in rational numbers, *c.* and *d.* in an equivalent decimal representation. The method of formation of each succeeding member is clearly shown. For example, in *a.* the numerator of the general member, a_n , consists of $(n + 1)$ digits, each *a* 9. The denominator consists of 3 followed by $(n + 1)$ zeros. If $n = 4$, the member is the number 99999/300,000.

a. 99/300, 999/3,000, 9999/30,000, 99999/300,000, ...

b. 102/300, 1,002/3,000, 10,002/30,000, 100,002/300,000, ...

c. 0.33, 0.333, 0.3333, 0.33333, ...

d. 0.34, 0.334, 0.3334, 0.33334, ...

Certain properties of the sequences must be noted. Each member of *a.*, after the first member, is greater in magnitude than the preceding member. Each member of *b.*, after the first member, is less than the preceding member. The decimal representation, *c.* and *d.*, show this clearly. Any member of class *a.* is less than 1/3. Any member of class *b.* is greater than 1/3. This is readily seen if we set up the following sequences, a' and b'. For convenience, let the members of *a.* be $a_1, a_2, a_3, \ldots$, with the general term a_n: the members of *b.* be $b_1, b_2, b_3, \ldots$, with the general term b_m.

$$a': 1/3 - a_2,\ 1/3 - a_2,\ 1/3 - a_3,\ \ldots$$
$$b': b_1 - 1/3,\ b_2 - 1/3,\ b_3 - 1/3,\ \ldots$$

For any given positive rational number δ, the positive numbers, $1/3, - a_n$ and $b_m - 1/3$, can be made less than δ by choosing respectively an n or m sufficiently large. This is merely another way of saying that we can approximate 1/3 as closely as desired. For example, let $\delta = 1/10^6$, i.e. the difference between 1/3 and a_n must be less than 0.000001. If $n = 4$, we have $(1/3 - 99999/300{,}000) > 1/10^6$. However, if $n = 5$, we have $(1/3 - 999{,}999/3{,}000{,}000) < 1/10^6$. If $m = 5$, $(1{,}000{,}002/3{,}000{,}000 - 1/3) < 1/10^6$. Since the difference between 1/3 and a_n cannot be zero, and the difference between b_m and 1/3 cannot be zero, with the exception of 1/3 the rational numbers are thus separated into two classes, *a.*, every member of which is less than 1/3 in magnitude, and *b.*, every member of which is greater in magnitude than 1/3. By our class selection, we therefore ensure that every member of *b*. is greater in magnitude then every member of *a*.. Every member, b_m, of *b* is called an upper bound for *a*.. However, there is *no least upper bound* in *b*. for *a*., since no matter what number *m* you select in *b*., the number $m + 1$ is such that b_{m+1} is less than b_m. The set of rational numbers is dense.

Arbitrarily we now include in class *b*. the rational number 1/3. Note that there is now a least upper bound in *b*. for *a*.. If we select *any* rational number *r*, less than 1/3, we can always find in *a*. a number *x* such that *x* is greater than *r*. The number 1/3 is the least upper bound in the set of rational numbers for the subset *a*. of rational numbers. More technically, the set of all rational numbers can be separated into two subsets or classes, *a*. and *b*., in such a manner that every rational number belongs either to *a*. or to *b*., but not to both. Neither set is empty and a_n in *a*. and b_m in *b*. implies a_n is less than b_m. There exists a cut number *c* such that a_n in *a*. implies a_n is less than or equal to *c* and b_m in *b*. implies *c* is less than or equal to b_m. The rational number 1/3 could have equivalently been included as a member of class *a*.. Then, 1/3 would have been the greatest upper bound in *a*. for *b*..

Technically, the separation of all rational numbers into two classes such that every number in one class is less than every number in the other class, is called a section or cut.

The rational number 1/3 has thus been defined unambiguously as a section or cut in the rational numbers. In a similar manner any other rational number can be exactly defined. For example, zero is the cut number between all positive rational numbers and zero, and the negative rational numbers.

This definition fundamentally depends upon the two converging sequences, *a*. and *b*., having the same limit, i.e. they define the same number. The two decimal sequences, *c*. and *d*., also converge to the same limit and define the same number, $0.\dot{3} = 1/3$. The difference between the corresponding members of *b*. and *a*., $b_1 - a_1$, $b_2 - a_2$, $b_3 - a_3$, ... forms a sequence which must also converge to a limit, zero. Two numbers are called equal if they differ by less than any preassigned constant, however small, i.e. there is no number between them. This is strikingly evidenced by the fact that $0.\dot{9}$ is equal to unity. Every rational number less than 1 will be exceeded ultimately by the decimal approximation 0.999...9. Every rational number greater than 1 will always exceed that decimal approximation. Hence, $0.\dot{9}$ is identically one. We note that there are consequently certain ambiguities in the decimal system since unity is the limit of the sequences 0.9, 0.99, 0.999, ..., a sequence whose general term can be represented as 0 decimal point followed by *n* digits, each 9. As *n* becomes larger and larger, the members of the sequence approach unity. The digit nine occurs endlessly after the decimal point and there can be no decimal representation of a number between it and unity.

The Square Root of Two

Since the set of rational numbers is dense and between two arbitrary rational numbers, r_1 less than r_2, a third, r_3, can always be found such that $r_1 < r_3 < r_2$, it should now be possible to find a number which multiplied by itself, yields the number two. Since $1^2 = 1$, less than two, and $2^2 = 4$, greater than two, we need only consider the rational numbers between 1 and 2. The square of any number less than 1 will be less than one, and the square of any number greater than two will be greater than four.

In investigating the numbers between 1 and 2 to find a number which, multiplied by itself, yields the number two, a methodical process will be used. For convenience a finite decimal representation will be used for the numbers.

A.	2.25	2.0164	2.002225	2.00024449	2.0000182084
B.	1.5	1.42	1.415	1.4143	1.41422
C.	1.4	1.41	1.414	1.4142	1.41421
D.	1.96	1.9881	1.999396	1.99996164	1.9999899241

A is a sequence every member of which is the square of the corresponding member of B. D is a sequence, every member of which is the square of the corresponding member of C. The first member, 1.4, of sequence C is chosen so that its square will be less than two, and so that it differs from the first member of B in the first digit after the decimal point. The square of every member of B is a number greater than two. The second members of the sequences B and C are selected so that they differ from each other only in the second digit after the decimal point while their squares are respectively greater than and less than two. The number whose square is two must lie between these two members of sequences B and C. The succeeding members of B and C are chosen in a similar manner. Hence, each succeeding member of B decreases in magnitude while each succeeding member of C increases in magnitude. The number whose square is two always lies between two corresponding members of B and C. The further out we go in each sequence, the more closely do their members approach the square root of two, one from above and one from below. The nth members of both sequences will differ from each other in the nth digit after the decimal point. The nth member of either sequence gives an approximation of the square root of two accurate to the $(n-1)$th decimal place. For example, the fifth member of either sequence is an accurate approximation of the square root of two to the fourth decimal place.

If the square root of two is a rational number it can be represented either by a terminating decimal, or by a pure or recurring decimal. If it is represented by a terminating decimal, a member of sequence B, or of sequence C will eventually be reached, whose square is exactly two. If it is represented by a pure or mixed infinite recurring decimal, a member of each sequence will eventually reach periodic forms of the type $0.\dot{3}$, $0.\dot{0}\dot{9}$, or $0.\dot{1}4285\dot{7}$. The fact is that we can carry out this process endlessly without obtaining such a result. Only a more accurate approximation of the desired number will be obtained. There is no least member of B since we can always find a member, n, further out in the sequence, whose square exceeds two by an ever decreasing positive amount. $n^2 = 2 + \delta$, where δ is a small positive number. There is no greatest member of C since a member, m, can always be found whose square is less than two by an ever decreasing amount. $m^2 = 2 - \delta$; i.e. δ is never zero. However, this is merely a statement and not a proof.

The Square Root Of Two Is Not A Rational Number

Centuries ago Euclid proved that no rational number squared is equal to two. In the interpretation of Euclid's proof, algebraic symbolism will be used although that symbolism did not appear upon the scene until centuries after his death. First he assumed that there is a rational number, a/b, from which all factors common to both the numerator and the denominator have been removed, and that the square of this rational number is two.

$$(a/b)^2 = 2. \qquad a^2 = 2b^2.$$

If $a^2 = 2b^2$, then a^2 is an even integer, because it has a factor, 2, shown in the right member. If a^2 is an even integer, then a is an even integer. If a is an even integer, it is divisible by two. Let $a = 2m$. Then $(2m)^2 = 2b^2$. $4m^2 = 2b^2$. $2m^2 = b^2$. Now b^2 is an even integer since it has a factor two shown in the left member. Hence, b is even. However, it was assumed that a/b was a rational number with no factor common both to numerator and denominator. Now it is evident that it has two as a common factor. The logical conclusion is that there is no rational number whose square is two. (The proof could be continued and the factor two removed endlessly.) How, then, is such a number to be defined?

The $\sqrt{2}$ Defined As A Section Or Cut

While it has been proved that the equation $a^2 = 2b^2$ has no solution, a and b integers, it has also been shown that all rational numbers, a/b, can be separated into two classes such that if $a^2 > 2b^2$, a/b is in the upper class, B, and if $a^2 < 2b^2$, a/b is in the lower class, C. For example, $1.415 = 1415/1000$. $(1415)^2 > (2)(10^3)$. As we go out endlessly in each sequence the members of class B will approach in magnitude constantly closer to the members of class C. We can therefore find a number which multiplied by itself will approximate as closely as we desire to the number two. Hence, for actual purposes of measurement the rational number system is adequate. In fact, the number $\sqrt{2}$ is *often used as if it were* a definite rational number. For example we consult a table of squares, cubes, and roots in a handbook. For $n = 2$ we find under $\sqrt{n}$, 1.414214. Actually all that this value means is that a number is selected from the lower class of the rationals as a satisfactory approximation.

While we were unable to find in either sequence a number which, multiplied by itself, yielded the number two, only by Euclid's proof do we logically know that no member of either class, B or C, will ever be the square root of two. We have an infinite number of rational numbers between 1 and 2, yet not one of them, multiplied by itself, yields the number two. This is astounding but logically unassailable. The set of rational numbers has, however, been separated into two classes B and C. They are infinite sequences, the squares of whose members approach closer and closer to two, one from above, the other from below. We cannot call them converging sequences since there is no number, $\sqrt{2}$, in the set of rational numbers. However, this difficulty is easily surmounted. A number, $\sqrt{2}$, is invented which has this unique property, $(\sqrt{2})(\sqrt{2}) = 2$. This number is the limit of each of these sequences. It is defined as a cut or section in the rational numbers, or as a limit of a sequence of rational numbers. Since $\sqrt{2}$ is not a rational number, *the number system is enlarged*, and called the set of *real numbers*, of which the rational numbers are a subset. In the set of rational numbers there is no least upper bound for the subset C. In the set of real numbers, R, there is a least upper bound, $\sqrt{2}$, for the subset C of set R. Defined as a cut or section, the rational number, a/b must belong to one or the other of the two classes of rational numbers. The number $\sqrt{2}$ is a member of neither class of rational numbers. A number has been invented which will do what we want it to do, as expressed by its unique property. In fact, the enlargement of the number system has been motivated from the beginning by this desire.

[Technically the idea that an irrational number must be the limit of a set of ratios is avoided as a logically unjustifiable postulation. Hence real numbers are defined, of which some will be rational and some will be irrational and the difference between them depends on the concept of cuts. However, the real number which corresponds to the rational number, a/b , though logically distinct from a/b, has no properties which differ from those of a/b. Hence, it is usually denoted by the same symbol. This is quite analogous to using a to denote both the integer a and the rational number $a/1$.]

Irrational Numbers

The fact that the equation $a^2 = 2b^2$ is not solvable, a and b integers, is not a special property of two. Let $x^2 = a/b$, a/b an arbitrary positive rational number in its lowest terms. Unless a and b are both perfect squares, i.e. the square of an integer, x is not a rational number. The equation $a^2 = pb^2$, p a prime number, is not solvable, a and b integers, and if we let $a^n = pb^n$, n and b integers, n greater then 1, $b \neq 0$, there is no integer a which satisfies the equations. Since any integer greater than unity is either a prime,or expressible as a product of primes, it is quite easy to believe that the square root of many integers is not a rational number. The square root of two is merely the simplest example.

Fleix Klein, the german mathematician, has left us an interesting derivation. He stated that the word "irrational" is without doubt the translation into latin of the greek "αλογοζ." The greek word, however, meant "inexpressible", and implied that the new numbers could not, like rational numbers, be expressed by the ratio of two whole numbers. The misunderstanding put upon the latin "ratio", that it could only convey the meaning "reason", gave to "irrational" the meaning "unreasonable", which of course, is a misnomer.

However, mathematically it is possible to express every irrational number as a limit of a sequence of rational numbers. Toward the end of the 16th century decimals were introduced. It was then found that rational numbers could be expressed as finite and infinite decimals. The infinite decimals were always periodic or recurring. It is easy to set up decimals which are infinite and a periodic. If we assume that all these decimals are numbers, then this subset of decimals constitutes the irrational numbers. However, the proof that a given number, such as π, is an irrational number is not always as simple as that for $\sqrt{2}$. The set of decimal numbers differs from the set of all rational numbers, or from the set of all irrational numbers, in that it is continuous and not discrete.

As mentioned previously, the entire set of rational and irrational numbers is called the set of real numbers. The word "real" is merely a name for the set of numbers which we have been discussing. *It has no other significance.* We might have the horrendous thought that if irrationals were further operated upon, a class of hyperirrationals might emerge, and perhaps so on, endlessly. This is not the case. The next enlargement of the real number system to the complex number system is such that the postulates of signs and order are no longer valid. The real number system is thus enlarged in certain respects and restricted in other respects.

The Postulate of Continuity.

The linearly ordered dense set of rational numbers has become continuous by postulating the existence, as numbers, of all the limits of *convergent sequences* (not defined in this article) of the rational numbers, i.e. there always exists a cut number. Continuity is the property which characterizes the set of real numbers. The real number system satisfies the postulate: any nonempty subset of the real numbers, which has an upper bound in the real number set, has a least upper bound in the set.

We shall illustrate this by using the sequences which define the square root of two. R is the set of real numbers, which includes $\sqrt{2}$. Since the rational numbers are a subset of the real numbers, then C is a nonempty subset of R. Technically a least upper bound in R for a subset C is defined in this manner. A number, r, in R is the least upper bound for C if there is no smaller upper bound, i.e. for any number, r_1 in R, such that r_1 is less than r, there is at least one number, y, in C such that r_1 is less than y. Note how the $\sqrt{2}$ fulfills the conditions for a least upper bound in R for the subset C. If r_1 is less than $\sqrt{2}$, no matter what number is chosen for r_1, we can, by going out far enough in the sequence C find there a number, y, such that y is greater than r_1. Only the number $\sqrt{2}$ will satisfy the conditions for the least upper bound in R of the subset, C. Of course equivalently the number 2 is the greatest lower bound in R of the subset B.

From the practical standpoint of measurements the introduction of rationals after integers is far more important than the further extension of rationals to real numbers. To include between two successive integers an endless number of rationals enables us always to obtain as accurate an approximation as desired. The rational number system is not sufficient for the purposes of geometry, algebra, calculus, and mathematical analysis. From a theoretical standpoint the irrational numbers are very essential since they provide numbers possessing certain properties. If these numbers did not exist, many generalizations could not be made. For example, the existence of certain limits such as π and epsilon depends upon the irrational numbers. In algebra the inequality symbolized by $(a + b) \geq \sqrt{ab}$, a and b positive integers, would be quite limited in scope since the square root of the product of two positive integers, i.e. their geometric mean, is generally an irrational number.

Operations Upon Irrational Numbers

Why can we operate upon these irrational numbers as we do upon rational numbers? This is a very important question. The answer is: we can enclose these irrational numbers between ever narrowing rational limits, and perform upon these rational limits the desired operations.

The result will also be enclosed between ever narrowing limits. As an example, separate all rational numbers, a/b, into two classes such that $a^2 < 5b^2$ constitutes the lower class, and $a^2 > 5b^2$ defines the upper class. The cut number is the square root of 5. Separate all rational numbers, c/d, into two classes. The upper class contains all rational numbers such that $c^2 > 7d^2$, and the lower class contains all rational numbers such that $c^2 < 7d^2$. The cut number is $\sqrt{7}$.

2.2361	belongs to the class $a^2 > 5b^2$	2.236	belongs to the class $a^2 < 5b^2$
2.646	belongs to the class $c^2 > 7d^2$	2.645	belongs to the class $c^2 < 7d^2$
4.8821		4.881	

The number 4.8821 belongs to the upper class, and the number 4.881 to the lower class, the cut number of which section defines the number $\sqrt{5} + \sqrt{7}$.

Conclusion

The following statements are not at all obvious but they have been shown to be logically unassailable. The rational numbers form a linearly ordered dense set. The irrational numbers form a linearly ordered dense set. Only the set of real numbers is continuous. It is a remarkable fact that all these numbers can be represented with only ten number symbols, 0,1,2,...,9.

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this Department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Indiana.

SOME NUMBERS RELATED TO THE BERNOULLI NUMBERS

Mike Rough - Junior, Van Nuys High School

We shall derive the formula

$$\sum_{v=1}^{n} (-1)^{v+1} v^s = (-1)^{n+1} \left(\beta_0 n^s + \frac{\beta_1}{1!} s n^{s-1} + \frac{\beta_2}{2!} s(s-1) n^{s-2} + \cdots\cdots + \frac{\beta_s}{s!} s(s-1) \cdots\cdots 2.1\right) + \beta_s \qquad (1);$$

where

$$\frac{1}{e^{-x}+1} = \sum_{v=0}^{\infty} \frac{\beta_v x^v}{v!} \qquad (2).$$

* * * * * * *

Sum the identity

$$(-1)^{v+1} e^{vx} = (-1)^{v+1} + (-1)^{v+1} v \frac{x}{1!} (-1)^{v+1} v^2 \frac{x^2}{2!} + \cdots\cdots + (-1)^{v+1} v^s \frac{x^s}{s!} + \cdots\cdots \qquad (3)$$

from $v = 1$ to $v = n$. The quantity on the left will then be a geometic progression having the sum $\dfrac{(-1)^{n+1} e^{nx} + 1.}{e^{-x}+1}$.

$$\frac{(-1)^{n+1} e^{nx} + 1}{e^{-x}+1} = \sum_{v=1}^{n} (-1)^{v+1} + \frac{x}{1!} \sum_{v=1}^{n} (-1)^{v+1} v \cdots\cdots + \cdots\cdots + \frac{x^s}{s!} \sum_{v=1}^{n} (-1)^{v+1} v^s + \cdots\cdots$$

$$\cdots\cdots \qquad (4).$$

However

$$(-1)^{n+1} e^{nx} + 1 = 1 + (-1)^{n+1} \left(1 + \frac{nx}{1!} + \frac{n^2x^2}{2!} + \dots\dots + \dots\dots + \frac{n^s x^s}{s!} + \dots\dots\right) \qquad (5).$$

Using Cauchy's rule to multiply (5) by (2) we obtain

$$\frac{(-1)^{n+1} e^{nx} + 1}{e^{-x} + 1} = \left\{(-1)^{n+1}\beta_0 + \beta_0\right\} + \frac{x}{1!}\left\{(-1)^{n+1}(\beta_0 n + \frac{\beta_1}{1!} 2) + \beta_1\right\} + \dots\dots$$

$$+ \dots\dots + \frac{x^s}{s}\left\{(-1)^{n+1}(\beta_0 n^s + \frac{\beta_1}{1} sn^{s-1} + \dots\dots + \frac{\beta_s}{s} s(s-1)\dots\dots 2.1) + \beta_s\right\}$$

$$+ \dots\dots \qquad (6).$$

Upon substituting this in (4) and equating coefficients of like powers we obtain (1).

By summing the expansion for e^{vx} from $v = 0$ to $v = n - 1$ we could have derived Bernoulli's formula:

$$\sum_{v=1}^{n-1} v^s = \frac{B_0}{s+1} n^{s+1} + B_1 n^s + \frac{B_2}{s} sn^{s-1}$$

$$+ \frac{B_3}{3!} s(s-1)n^{s-2} + \dots\dots + \frac{B_s}{s!} s(s-1) \dots\dots 2.1n \qquad (7);$$

where the B's are the Bernoulli numbers, defined by the expansion

$$\frac{x}{e^x - 1} = \sum_{v=0}^{\infty} \frac{B_v}{v} x^v \qquad (8).$$

The β's possess some interesting properties:

Corollary I:

$$2\beta_i = 1 - \sum_{v=0}^{i-1} \beta_i \binom{s}{v} \qquad (9).$$

Proof: Set $n = 1$ in (1) and (9) follows immediately.

The first nine of the numbers are:

$$\beta_0 = \frac{1}{2};\ \beta_1 = \frac{1}{4};\ \beta_2 = 0;\ \beta_3 = -\frac{1}{8};\ \beta_4 = 0;\ \beta_5 = \frac{1}{4};\ \beta_6 = 0;$$

$$\beta_7 = -\frac{17}{16}, \ \beta_8 = 0; \ \beta_9 = \frac{31}{4}.$$

Corollary II: $\beta_{2i} = 0$ for $i \geq 1$ (10).

Proof: From (9) $\beta_0 = 1/2$; substituting this in (2):

$$\frac{1}{e^{-x}\ 1} - \frac{1}{2} \equiv \frac{1}{2}\ \frac{1-e^{-x}}{1+e^{-x}} = \sum_{v=1}^{\infty} \frac{\beta_v}{v!} x^v \qquad (11);$$

The quantity on the left is an odd function; hence $\beta_{2i} = 0$.

Corollary III: $\beta_{2i-1} = \frac{P}{Q}$ where P is an integer and Q is a power of 2(i.e. $\beta_{2i-1} = \frac{P}{2^m}$; $m \leq 2i - 1$.).

Proof: This follows immediately from (9) by mathematical induction.

Corollary IV: $B_{2i} = \frac{2i}{2^{2i}-1}\beta_{2i-1}$ (12)*

Proof: It is known that

$$\tan x = \sum_{v=1}^{\infty} (-1)^{v-1} \frac{2^{2v}(2^{2v}-1)}{(2v)!} B_{2v} x^{2v-1} \qquad (13).$$

Also

$$\frac{1}{2}\frac{1-e^{-x}}{1+e^{-x}} \equiv \frac{1}{2}\ \frac{e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}}{e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}} \equiv \frac{1}{2}\tanh\frac{1}{2}x \qquad (14);$$

and since $\tanh\frac{1}{2}ix = i\tan\frac{1}{2}x$ it follows from (10) and (11) that

$$\tan x = \sum_{v=1}^{\infty} (-1)^{v-1} \frac{2^{2v-1}}{(2\ \ -1)!}\ \beta_{2v-1} x^{2v-1} \qquad (15).$$

The result follows upon comparing (13) and (15).

Combining corollary III with (12) gives an interesting expression for the B's. Thus

$$B_{2i} \equiv \frac{2iP}{2^{2m}(2^{2i}-1)} \qquad (16).$$

In using (9) we need only operate with fractions having denominators as powers of 2(cf. corollary III). Because of this, (9) and (12) affords a convenient method for calculating the Bernoulli numbers.

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* The B's are the Bernoulli numbers mentioned earlier.

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

ON THE FIBONACCI SEQUENCE

Dear Dr. James:

In his article "A Type of Periodicity for Fibonacci Numbers," (Math. Magazine, V. 28, no. 3), Vern Hoggart raises the question of whether or not three consecutive members F_n, F_{n+1}, F_{n+2} of the Febonacci sequence can be perfect squares. It is easy to answer this question in the negative, and in fact to prove that $F_n F_{n+2}$ is never a square $(n > 0)$. For

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}$$

as is well known, and if $F_n F_{n+2} = x^2$, this implies at once that $x^2 = 1$, $F_{n+1} = 0$ or $x^2 = 0$, $F_{n+1} = 1$. Both of these are impossible for $n > 0$.

Sincerely yours,

Mr. Basil Gordon

* * * *

Arithmetic For Teacher-Training Classes, by E.H.Taylor, formerly Eastern Illinois State College and C.N.Mills, Illinois State Normal University. Henry Holt and Company, 438 pp., $4.25.

A text for prespective teachers, this book is based on the assumption that a complete mastery of subject matter is of prime importance in teaching arithmetic effectively. The book therefore begins with simple arithmetic computations and proceeds logically and clearly to present the whole range of the subject. The student is taught by the same methods which he in turn will find most successful in teaching his own pupils

A special feature of the Fourth Edition of this widely used text is the emphasis on teaching the meanings of fundamental operations with integers, fractions, and decimals. The problems have been brought up to date and there are numerous graded exercises. Pertinent historical and scientific notes add to the student's knowledge of the development of arithmetic and its place in our culture.

Henry Holt and Company

Advanced Calculus. By Louis Brand, John Wiley & Sons, 440 Fourth Ave., New York 16, New York, 574 pp., $8.50.

A systematic treatment of classical analysis, *Advanced Calculus* by Louis Brand was published in June. This new book ranges from the development of the number system to the functions of the real and complex variable.

With special attention given to motivation of an orderly, logical approach, Dr. Brand opens with the structure of the system of real and complex variables. He then deals at length with the conversion of sequences and series, introducing some of the basic concepts of analysis in their simplest setting, through functions of an integral variable. Chapters follow on functions of a real variable and several variables.

Following is an introduction to vectors and the important differential invariants, gradient, divergence, and rotation. Other chapters are concerned with the definite integral, improper integrals, line and multiple integrals, uniform convergence, and functions of a complex variable. The final chapter covers the Fourier series and concludes with Gibb's phenomenon.

Richard Cook

Machine Translation of Languages. Published by John Wiley & Sons and The Technology Press of the Mass. Institute of Technology, 1955 243 pp., $6.00.

It is now very possible that machines will be able to restore the "mythical situation" of a world where language is no longer a barrier to communication. The origins, evolution, and new directions of this highly feasible language medium are explored in this new book. It contains fourteen essays by top-ranking men now devoting their insight and technical skills to this very field.

In his foreword entitled "The New Tower," Warren Weaver compares and contrasts the goals of machine translation and the Tower of Babel. "The hopes for this new development are, one can believe, so reasonable and limited, that this new tower (of anti-Babel) will not fail through arrogance," Weaver states. "No reasonable person thinks that a machine translation can ever achieve elegance and style. Pushkin need not shudder. And the kinds of questions that enter in connection with the translation of the Bible will continue to require at least fifty learned men. But, on the other hand, there is now reasonable hope that the new tower can be a little higher than at first seemed possible."

Not for Heaven, as Weaver points out, but certainly for earth, machine translation is already on its way towards achieving reasonable success. Its already detailed history is outlined by A. Donald Booth and William N. Locke, editors of the volume. Weaver's earlier well-known contribution, "Translation," is then reprinted in its entirety. Some methods of mechanized translation are related by R. H. Richens and A. Donald Booth, and Anthony G. Oettinger discusses the design of an automatic Russian-English technical dictionary. This is followed by a preliminary study of Russian, written by Kenneth E. Harper. Some problems of the "word" are the concern of William E. Bull, Charles Africa, and Daniel Teichroew.

A. Donald Booth appears again for a discussion on storage devices. Following is Leon E. Dostert's consideration of the Georgetown-I.B.M. Experiment and Erwin Reifler's study of the mechanical determination of meaning. Model English is Stuart C. Dodd's concern. A practical development problem is covered by James W. Perry, idioms by Yehoshua Bar-Hillel, and some logical concepts for syntax by Luitgard and Alexander Wundheiler. Victor H. Yngve concludes the volume with syntax and the problem of multiple meaning.

With the acknowledged value of language simplification to translators, linguists, diplomats, and businessmen, it was logical that the seventeen authorities who shared the writing of this book be concerned with practical considerations. All work closely with the most rigorous concepts of mathematics, electronics, and various languages. William N. Locke is Head of the Department of Modern Languages at the Massachusetts Institute of Technology, and A. Donald Booth is Director of the Birbeck College Computation Laboratory of London.

Richard Cook

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor.
Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

PROPOSALS

250. *Proposed by V. C. Harris, San Diego State College, California.*

In how many different ways can the game of bowling take place? Consider only the number of pins knocked down by each ball, not the particular position of the pins.

251. *Proposed by George U. Brauer, University of Minnesota.*

Find all integers a and b such that $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ have integral roots.

252. *Proposed by Donald M. Brown, Willow Run Research Center.*

Given a square with vertices $ABCD$ with sides of unit length, and a point E on side AB such that $AE = 1/n$ units. Devise a simple geometric construction to locate the point F such that $AF = 1/(n+1)$ units.

253. *Proposed by M. S. Klamkin and C. H. Pearsall Jr., Polytechnic Institute of Brooklyn.*

In Ripley's (New) "Believe It Or Not" the following statement appears (p. 207). "The Persistent Number 526, 315, 789, 473, 684, 210 may be multiplied by any number. The original digits will always reappear in the result." Show that this statement is not correct.

254. *Proposed by Barney Bissinger, Lebanon Valley College.*

If $S_1 = k \geq 1$ and $S_{n+1} = 4S_n^3 - 3S_n$ for $n \geq 1$ find $\lim\limits_{n\to\infty} \dfrac{S_{n+1}}{\prod_1^n (4S_i^2 - 1)}$

255. *Proposed by Leon Bankoff, Los Angeles, California.*

Two orthogonal circles, centers at A and B, intersect in P and

Q. A line through P cuts circle A in M and circle B in N. S is the mid-point of MN. Show by elementary geometry that the nine-point circles of triangles ABP and ABS are equal.

256. *Proposed by Maïmouna Edy, Hull, P, Q. Canada.*

Let $u = u(t)$ be a p- times differentiable function from an interval (a,b) to an N- dimensional real vector space. The successive derivatives of u are denoted by $u', u'', \ldots, u^{(p)}!$ Let the first $(p - 1)$ of these derivatives be linearly independent while the first p of them are linearly dependent for all t in (a,b). Show that u belongs to a fixed subspace of dimension p.

SOLUTIONS

Late Solutions

225. *N. Shklov, University of Saskatchewan.*

224. *M. G. Probhakar, Mysore City, India.*

The Perpendicular To A Median

229. [March 1955] *Proposed by Chih-yi Wang, University of Minnesota.*

Let AD be a median of a triangle ABC. If CE is perpendicular to AD and angle ACE equals angle ABC, prove, geometrically that either $AB = AC$ or angle BAC is a right angle.

I. *Solution by V. D. Gotchak, Atlanta University, Georgia.* If the point E. coincides with D, this implies that $AB = AC$ and conversely. If angle B is acute and E differs from D let T be the point where CE meets AB. The angle ATC = angle C whether E is inside or outside the triangle. Hence angle $DAB = 90° - C$. Thru B draw the perpendicular to AB meeting the median at L. Then angle ALB = angle C so the points A, B, C and L are concyclic with AL as a diameter. Now diameter AL bisects chord BC so D is the center of the circle $ABCL$. Thus angle A is inscribed in a semicircle. Therefore angle A is a right angle.

II. *Solution by M. N. Goplan, Mysore, India.* Let the perpendicular from A to BC meet BC at M.

We are given that $\angle ACE = \angle ABC$

Therefore $\angle BAM = \angle DAC = 90° - \angle B$

$$\angle DAM = \angle DAC - \angle MAC$$

$$= (90° - \angle B) - (90° - \angle C)$$

$$= \angle C / -\psi \angle B$$

If S and O are the circum-centre and the orthocentre of triangle ABC respectively, then

$$\angle SAO = \angle C - \angle B$$

But S lies on the perpendicular bisector of BC

Therefore S must lie on AD

Therefore AD must be perpendicular to BC in which case

$$AB = AC$$

S may coincide with D. Then $AD = BD = DC$,

D being the mid-point of BC

Therefore $\angle A = 90°$

Also solved by Huseyin Demir, Zonguldak, Turkey; Alan Wayne, Cooper Union School of Engineering, New York; Hazel S. Wilson, Jacksonville State College, Alabama; and the proposer (two solutions).

A Factored Product

230. [March 1955] *Proposed by John M. Howell, Los Angeles City College.*

Prove
$$\sum_{x=0}^{2n} (-1)^x \binom{2n}{x}^2 = (-1)^n \binom{2n}{n}$$

I. *Solution by H. M. Feldman, St. Louis Public Schools, Missouri.* If we expand both members of the identity $(1-a)^{2n}(1+a)^{2n} = (1-a^2)^{2n}$ we obtain

$$\sum_{x=0}^{2n} (-1)^x \binom{2n}{x} a^x \sum_{y=0}^{2n} (-1)^y \binom{2n}{y} a^{2n-y} = \sum_{x=0}^{2n} (-1)^x \binom{2n}{x} (a^2)^x$$

Equating coefficients of the terms containing a^{2n} on the two sides of this equation gives the desired result.

II. *Solution by Earl D. Rainville, University of Michigan.* A fairly well known formula, probably due to Bateman, for the Legendre polynomial $P_m(y)$ is

$$(1) \qquad P_m(y) = 2^{-m} \sum_{x=0}^{m} \binom{m}{x}^2 (y-1)^x (y+1)^{m-x}$$

See, for instance, the Bateman Manuscript Project, Erdélyi, *Higher Transcendental Functions*, McGraw-Hill, 1953, vol. 2, p. 169, (12), and use $\alpha = \beta = 0$. Equation (1) follows at once from formula (7), page 262, of vol. 3, 1954, Erdélyi, *Higher Transcental Functions*; in their (7) form the product series on the left, then equate coefficients of t^n.

In (1), put $y = 0$ and $m = 2n$ to obtain

$$2^{2n} P_{2n}(0) = \sum_{x=0}^{2n} (-1)^x \binom{2n}{n}^2$$

From almost any book which touches on Legendre polynomials we obtain

$$2^{2n}P_{2n}(0) = (-1)^n \binom{2n}{n},$$

which completes the proof.

Also solved by Huseyin Demir, Zonguldak, Turkey; V. D. Gotchak, Atlanta University, Georgia; N. Shklov, University of Saskatchewan; Harry Siller, Far Rockaway, New York; E.P.. Starke, Rutgers, University; Chih-yi Wang, University of Minnesota and the proposer.

Palindromes In Progression

231. [March 1955] *Proposed by Leo Moser, University of Alberta.*

A number is called palindromic if it reads the same forward and backward, when written with the base 10. Find four palindromic primes in arithmetic progression less than 10^6.

Solution by E.. P.. Starke, Rutgers University. A four-or six-digit palindrome is necessarily divisible by 11, hence no prime. Consider three-digit palindromic primes. The first digit can not be 2,4,5,6, or 8, hence four in arithmetic progression cannot lie in different hundreds. Thus the common difference between successive terms must be 10 or 20 or 30. It cannot be 10 or 20 because one of three consecutive multiples of 10 or 20 must be divisible by 30, hence no prime. Then the middle digits will have to be 0,3,6,9. But 161, 303, 707, 909 are not primes.

The solution must be sought among five-digit numbers. Again, since the first digit must be 1,3,7, or 9, the four terms of the progression must lie in the same 10,000. It takes only a few minutes, with aid of a table of primes, to write down the palindromic primes within one of the possible 10,000's. If four of these numbers are in arithmetic progression so are the numbers composed of their three middle digits. Since there are only 18 to 26 numbers involved in each case, it is a simple matter to select the desired sets of four. We find

13931, 14741, 15551, 16361; 10301, 13331, 16361, 19391;
70607, 73637, 76667, 79697; 94049, 94349, 94649, 94949.

Also solved by Leon Bankoff, Los Angeles, California; Frank W. Saunders, Coker College, South Carolina and the proposer.

A Square Dissection

232. [March 1955] *Proposed by Leon Bankoff, Los Angeles, California.*

Employing a minimum number of operations with straight-edge and compasses, dissect a square into three segments whose areas are in the ratio 3:4:5.

Solution by Donald M. Brown, Willow Run Research Center, Michigan. Given the square $ABCD$ as shown in Figure 1

1. With D as center and DC as radius strike an arc immediately below D
2. Extend CD to intersect this arc at point E
3. Draw line BE intersecting AD at F
4. Draw AC intersecting BE at G

Then polygons ABF, BGC, and $FGCD$ have areas in ratio 3;4;5, respectively.

Proof- Let A be the origin for set of axes as shown in Figure 1 and let the side of the square be one unit long. From the construction it is clear that $FD = \frac{1}{2}BC$, or $FD = \frac{1}{2} = AF$ since the intercepts of line BE are $(\frac{1}{2},1)$, its equation is $x + 2y = 1$, while the equation of line AC is $y = x$. The intersection of these two lines is easily found to be $x = y = 1/3$. Hence the altitude of triangle BCG at G is 2/3. Therefore, the areas of polygons ABF, BGC, FGCD are $1(\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}$, $1(2/3)(\frac{1}{2}) = 1/3$, $1 - \frac{1}{4} - 1/3 = 5/12$, respectively, or 3/12, 4/12, 5/12, or their areas are in ratio 3:4:5, respectively, as required.

In two operations the best that can be done is to draw the 2 diagonals of the square. In three operations one can either draw the 2 diagonals and an arc from one vertex, with radius either equal to the side of the square or equal to half the diagonal, none of which will produce areas of required proportion. Thus it is clear that at least 1 additional point other than the 4 vertices is needed. Since this point cannot be chosen at random it must be located by two lines, an arc and a line, or 2 arcs. After this point is located, at least one line from it plus one other line will be needed to dissect the square. Hence it is clear that at least four operations are needed to make the required dissection. This completes the proof.

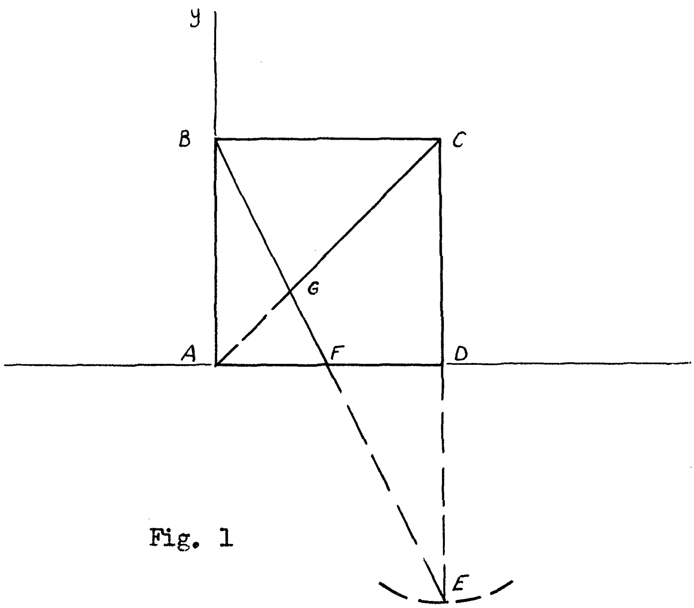

Fig. 1

Editor's note: Using Lemoine's criteria for Simplicity and Exactitude (see Daus, *College Geometry*, page 154) The above construction scores (S12, E8). The scores of the other solvers follow their names.

Also solved by Rafael T. Coffman (S18,E12). Richland, Washington; M. A. Kirchberg (S13,E8) Hopkins, Michigan; Ezra Shahn and John Simon (Jointly)(S29,E17), Bard College, New York; R. E. Shafer, (S20,E13) University of California at Berkeley; N. Shklov (S27,E17), University of Saskatchewan; E. P. Starke (S29,E18), Rutgers University and the proposer (S13,E8).

"Rabattments" of Altitudes

233. [March 1955] *Proposed by V. Thebault, Tennie, Sarthe, France.*

The points Q, P are the projections upon the sides C and B of a triangle ABC of the feet B', C' of the altitudes BB' and CC'. Show that, (a) PQ is parallel to BC, and the distance between the two parallel lines is equal to S/R, where S is the area and R is the circumradius of ABC. (b) The lines $B'Q$ and $C'P$ are perpendicular to each other and are equally inclined to the bisectors of the angle A.

Editor's note: Leon Bankoff learned through correspondence with V. Thebault that the word projection had been used to translate rabattment. Thus a better translation of Thebault's problem is:

The points Q, P are the intersections of the sides c and b of a triangle ABC with the arcs with centers B and C of radius BB' and CC' respectively, where B' and C' are the feet of the altitudes BB' and CC'. Show that, (a) PQ is parallel to BC, and the distance between the two parallel lines is equal to S/R, where S is the area and R is the circumradius of ABC. (b) The lines $B'Q$ and $C'P$ are perpendicular to each other and are equally inclined to the bisectors of the angle A.

Solution By Leon Bankoff, Los Angeles, California.

(a) From the similarity of right triangles $AC'C$ and ABB', and the consequent similarity of the isosceles triangles QBB' and $C'CP$, it follows that angles $B'QB$ and CPC' are equal. Hence Q, C', B', P are concyclic, and QP and $B'C'$ are anti-parallel with respect to AB and AC.

Since the circle on diameter BC passes through C' and B', $B'C'$ and BC are also anti-parallel with respect to AB and AC. It follows that QP and BC are parallel.

If d denotes the distance between QP and BC, and h_a the altitude from A upon BC, we have

$$d/h_a = BQ/AB = BB'/AB = \sin A = BC/2R$$

Hence

$$d = h_a\, BC/2R = S/R.$$

(b) In the similar triangles QBB' and $C'CP$, angles CPC' and $QB'B$ are equal. Since angles $QB'P$ and $QB'B$ are complementary,it follows that angles $QB'P$ and $B'PC$ are also complementary and that $QB'$is perpendicular to PC'.

The equality of angles $B'QB$ and CPC', shown above, implies that PC' and QB' are equally inclined to the sides of angle A and, consequently, to the internal and external bisectors of angle A.

Also solved by P.D.. Thomas, Eglin Air Force Base, Florida who recognized the kind of rotation that Thebault intended.

Squares In A Lattice

234. [March 1955] *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Given an m by n rectangular lattice containing mn points, find the total number of (a) squares, (b) rectangles having vertices at the points of the lattice. Consider $m \geq n$.

Solution by the proposer. We distinguish two kinds of squares. A square is an N- or L-square according as their sides are or are not parallel to the sides of the lattice.

Every L-square is inscribed in a unique N-square. By a $p \times p$ N-square we mean one having p points on each of its sides. In such a square are inscribed evidently $p-2$ L-squares. Including the N-square itself the number is $p-1$.

The number of $p \times p$ N-squares is easily seen to be $(m - p + 1)(n - p + 1)$. Hence the number of $p \times p$ N-squares together with L-squares inscribed in them is $(p - 1)(m - p + 1)(n - p + 1)$. Hence the required total number of squares is given by

$$N = \sum_{p=2}^{m} (p - 1)(m - p + 1)(n - p + 1)$$

$$= mn \sum (p - 1) - (m + n) \sum (p - 1)^2 + \sum (p - 1)^3$$

$$= mn \frac{n(n - 1)}{2} - (m + n)\frac{n(n - 1)(2n - 1)}{6} + \frac{n^2(n - 1)^2}{4}$$

$$= \frac{n(n - 1)}{12}\left[6mn - 2(m + n)(2n - 1) + 3n(n - 1)\right]$$

$$= n(n^2 - 1)(2m - n)/12.$$

No solution of the rectangular case has been received. Solutions restricting the squares and rectangles to those with sides parallel to the lines of lattice points were received from *Julian H. Braun, White Sands Proving Ground and E. P. Starke, Rutgers University.* Braun noted that the restricted case was a variation of Problem E 1127 of the *American Mathematical Monthly.*

Equidistant Points

235. [March 1955] *Proposed by N. Shklov, University of Saskatchewan.*

Find the coordinates of the point P whose distances from each of the points $(1/y_1, y_1)$; $(1/y_2, y_2)$ and $(y_1 y_2 y_3, \frac{1}{y_1 y_2 y_3})$ are equal.

Solution by Harry Siller, Far Rockaway New York. Simplifying, find the coordinates (x,y) of the point equidistant from $(1/a, a)$, $(1/b, b)$ and $(1/c, c)$. From the first two points we have

$$(x - 1/a)^2 + (y - a)^2 = (x - 1/b)^2 + (y - b)^2$$

which reduces to $2abx - 2a^2b^2y = (a + b)(1 - a^2b^2)$ (1)
From symmetry using the first and third points we have

$$2acx - 2a^2c^2y = (a + c)(1 - a^2c^2) \qquad (2)$$

Solving equations (1) and (2) for x and y we obtain

$$x = \frac{ab + bc + ca + a^2b^2c^2}{2abc}$$

and $y = \dfrac{1 + abc(a + b \pm c)}{2abc}$. Replacing a, b and c by y_1, y_2, and $\dfrac{1}{y_1 y_2 y_3}$ respectively provides the solution

$$x = \frac{y_1^2 y_2^2 y_3^2 + y_1 y_2 + y_2 y_3 + y_3 y_1}{2 y_1 y_2 y_3} \qquad (3)$$

$$y = \frac{1 + y_1 y_2 y_3 (y_1 + y_2 + y_3)}{2 y_1 y_2 y_3} \qquad (4)$$

It should be noted that the three given points lie on the equilateral hyperbola $xy = 1$. Because of symmetry of (3) and (4) with respect to $y_1 y_2$ and y_3, it follows that $(1/y_3, y_3)$ lies on $xy = 1$ and also on the circle passing through the given points. Since any circle intersects the hyperbola in at most four points this implies that the coordinates of points of intersection of any circle with the hyperbola $xy = 1$ are of the form:

$$(1/y_1, y_1),\ (1/y_2, y_2),\ (1/y_3, y_3) \text{ and } \left(y_1 y_2 y_3, \frac{1}{y_1 y_2 y_3}\right).$$

Also solved by Huseyin Demir, Zonguldak, Turkey; M. N. Gopalan, Mysore, India; Robert E. Shafer, University of California at Berkely; Viktors Linis, University of Ottawa; E. P. Starke, Rutgers University; Chih-yi Wang, University of Minnesota; Hazel S. Wilson, Jacksonville State College, Alabama and the proposer. One anonymous solution was received.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, butwhich with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 152. Prove that the average of the square of a velocity is greater than or equal to the square of the average velocity. [*Submitted by M. S. Klamkin*].

Q 153. A matrix A has a property that if any one of its columns is replaced by a column of 1's the determinant of the resulting matrix is zero. Prove that the determinant of A is zero. [*Submitted by Albert Wilansky*].

Q 154. Show that $V_n/S_n = r/n$ where V_n and S_n are the volume and surface of an n-dimensional sphere. [*Submitted by M. S. Klamkin*].

Q 155. Prove that the radius of the inscribed circle in a right triangle is an integer if the sides of the triangle are integers. [*Submitted by Fred G. Elston*].

Q 156. Solve $4x^3 - 6x^2 + 4x - 1 = 0$. [*Submitted by M.S.Klamkin*].

Q 157. Evaluate $\cos 24° + \cos 48° + \cos 96° + \cos 192°$. [*Submitted by Norman Anning*].

Q 158. Find the area of the ellipse $4x^2 + 2\sqrt{3}\,xy + 2y^2 = 5$. [*Submitted by M.S.Klamkin*].

ANSWERS

A 152. To prove $V^2 \geq (V)^2$ we have that

$$\frac{\int_a^b V^2 dp}{\int_a^b dp} \geq \left[\frac{\int_a^b V dp}{\int_a^b dp}\right]^2$$

follows from the Cauchy-Schwartz Inequality.

A 153. Let Y be the column vector of 1's. Suppose the determinant of A is not zero. Then we may apply Cramer's rule to solve the system of equations $AX = Y$. The hypothesis gives $X = 0$ so $A \cdot 0 = Y$ which is a contradiction. The same applies if Y is any non zero column vector.

A 154. From similar figures it follows that $V_n = K_1 r^n$ and $S_n = K_2 r^{n-1}$. But by dividing the sphere up into concentric spheres $dV_n = S_n dr$ so that $K_1 n = K_2$ and $\frac{V_n}{S_n} = \frac{K_1 r}{K_2} = \frac{r}{n}$.

A 155. Let A be the right angle. The half angle formula gives $\tan A/2 = r/(s-a)$ so $1 = r/(s-a)$ or $r = s-a$. As s and a are integers r is also.

A 156. The given equation is equivalent to $(x-1)^4 = x^4$ or $(2x-1)[(x-1)^2 + x^2] = 0$ which has roots $x = 1/2$, $(1 \pm i)/2$.

A 157. The answer is ½. This becomes evident if one begins by projecting the sides of a regular pentagon on a suitably chosen straight line and noting that $\cos 24° + \cos 96° + \cos 168° + \cos 240° + \cos 312° = 0$.

A 158. Rotate the ellipse into the form $Ax^2 + Cy^2 = 5$. Then $A + C = 6$ and $-4AC = (2\sqrt{3})^2 - 4(4)(2)$. That is $A = 5$ and $C = 1$ or $A = 1$ and $C = 5$. Thus the area is $\pi\sqrt{5}$.